Energy decay of solutions to a wave equation with a dynamic boundary dissipation of fractional derivative type

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Considered system

In this talk, we consider an initial boundary value problem for the linear wave equation reading as

$$(P) \qquad \varphi_{tt}(x,t) - \varphi_{xx}(x,t) = 0 \text{ in }]0,L[\times]0,+\infty[,$$

where $(x,t) \in (0,L) \times (0,+\infty)$. This system is subject to the boundary conditions

$$\begin{array}{ll} \varphi(0,t) = 0, & \text{in } (0,+\infty) \\ m\varphi_{tt}(L,t) + \varphi_x(L,t) = -\gamma \partial_t^{\alpha,\eta}\varphi(L,t) & \text{in } (0,+\infty) \end{array}$$

where $m > 0$ and $\gamma > 0$.
The problem (P) describes the motion of a pinched vibration cable with tip mass $m > 0$.

Considered system

The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α with respect to the time variable. It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \ \ \eta \geq 0.$$

The system is finally completed with initial conditions

$$\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x),$$

where the initial data (φ_0, φ_1) belong to a suitable Sobolev space.

Physical interpretations

The boundary feedback under the consideration are of fractional type and are described by the fractional derivatives

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \ \ \eta \geq 0.$$

The order of our derivatives is between 0 and 1. Very little attention has been paid to this type of feedback. In addition to being nonlocal, fractional derivatives involve singular and nonintegrable kernels $(t^{-\alpha}, 0 < \alpha < 1)$. This leads to substantial mathematical difficulties since all the previous methods developed for convolution terms with regular and/or integrable kernels are no longer valid.
It has been shown that, as ∂_t, the fractional derivative ∂_t^{α} forces the system to become dissipative and the solution to approach the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to reduce the vibrations.

Physical interpretations

Boundary dissipations of fractional order or, in general, of convolution type are not only important from the theoretical point of view but also for applications. They naturally arise in physical, chemical, biological, ecological phenomena . They are used to describe memory and hereditary properties of various materials and processes. For example, in viscoelasticity, see for example the early work of

- **R. L. Bagley and P. J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, J. Rheology. 27 (1983), 201ñ210**.
- **R. L. Bagley and P. J. Torvik, On the appearance of the fractional derivative in the behavior of real material, J. Appl. Mech. 51 (1983), 294-298.**

In our case, the fractional dissipations may describe an active boundary viscoelastic damper designed for the purpose of reducing the vibrations.

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Augmented model

To reformulate the model (P) into an augmented system, we need the following claims.

Theorem

Let μ be the function :

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1.$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0,$$

$$\phi(\xi, 0) = 0,$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t)\, d\xi$$

is given by

$$O(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} e^{-\eta(t-\tau)} U(\tau)\, d\tau.$$

System (P) may be recast into the augmented model :

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = 0, \\ \partial_t \phi(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) - \varphi_t(L, t)\mu(\xi) = 0, \\ \varphi(0, t) = 0, \\ m\varphi_{tt}(L, t) + \varphi_x(L, t) = -\gamma(\pi)^{-1} \sin(\alpha\pi) \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t)\, d\xi, \\ \varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x). \end{cases} \tag{P'}$$

Energy function

$$E(t) = \frac{1}{2}\|\varphi_t\|_2^2 + \frac{1}{2}\|\varphi_x\|_2^2 + \frac{m}{2}|\varphi_t(L,t)|^2 + \frac{\gamma}{2}(\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}(\phi(\xi,t))^2\,d\xi.$$

Dissipation of (P)

$$E'(t) = -(\pi)^{-1}\sin(\alpha\pi)\gamma\int_{-\infty}^{+\infty}(\xi^2+\eta)(\phi(\xi,t))^2\,d\xi \leq 0.$$

We have $E' \leq 0$, and then the system (P) is dissipative, where the dissipation is guaranteed by the finite memory term.
If $\gamma = 0$ (no memory term in (P)), then $E = E(0)$, and therefore (P) is conservative.

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Well-posedness

Let $U = (\varphi, \varphi_t, \phi, v)^T, v = \varphi_t(L)$. (P') is equivalent to

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (\varphi_0, \varphi_1, \phi_0, v_0), \end{cases} \tag{1}$$

$$\mathcal{A}\begin{pmatrix} \varphi \\ u \\ \phi \\ v \end{pmatrix} = \begin{pmatrix} u \\ \varphi_{xx} \\ -(\xi^2+\eta)\phi + u(L)\mu(\xi) \\ -\frac{1}{m}\varphi_x(L) - \frac{\zeta}{m}\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi \end{pmatrix}$$

Well-posedness

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (\varphi, u, \phi, v)^T \text{ in } \mathcal{H} : \varphi \in H^2(0,L) \cap H_L^1(0,L), u \in H_L^1(0,L), v \in \mathbb{C}, \\ -(\xi^2 + \eta)\phi + u(L)\mu(\xi) \in L^2(-\infty, +\infty), u(L) = v, \\ |\xi|\phi \in L^2(-\infty, +\infty) \end{array} \right\} \tag{2}$$

where, the energy space $\mathcal{H}$ is defined as

$$\mathcal{H} = H_L^1(0,L) \times L^2(0,L) \times L^2(-\infty, +\infty) \times \mathbb{C}.$$

For $U = (\varphi, u, \phi, v)^T, \overline{U} = (\overline{\varphi}, \overline{u}, \overline{\phi}, \overline{v})^T$, we define the following inner product in $\mathcal{H}$

$$\langle U, \overline{U} \rangle_{\mathcal{H}} = \int_0^L (u\overline{u} + \varphi_x \overline{\varphi}_x)\, dx + \zeta \int_{-\infty}^{+\infty} \phi\overline{\phi}\, d\xi + mv\overline{v}.$$

The operator $\mathcal{A}$ generates a C_0- semigroup in $\mathcal{H}$. In this step, we prove that the operator $\mathcal{A}$ is dissipative. Let $U = (\varphi, u, \phi, v)^T$. Using the fact that

$$E(t) = \frac{1}{2}\|U\|_{\mathcal{H}}^2, \tag{3}$$

we get

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi(\xi))^2 \, d\xi \tag{4}$$

Consequently, the operator $\mathcal{A}$ is dissipative. Now, we will prove that the operator $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. For this purpose, let $(f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, we seek $U = (\varphi, u, \phi, v)^T \in D(\mathcal{A})$ solution of the following system of equations

Well-posedness

$$\begin{cases} \lambda\varphi - u = f_1, \\ \lambda u - \varphi_{xx} = f_2, \\ \lambda\phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = f_3, \\ \lambda v + \frac{1}{m}\varphi_x(L) + \frac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = f_4. \end{cases} \tag{5}$$

Problem (5) is equivalent to the problem

$$a(\varphi, w) = L(w) \tag{6}$$

where the bilinear form $a : H_L^1(0, L) \times H_L^1(0, L) \to \mathbb{R}$ and the linear form $L : H_L^1(0, L) \to \mathbb{R}$ are defined by

Well-posedness

$$a(\varphi, w) = \int_0^L (\lambda^2 \varphi w + \varphi_x w_x)\, dx + \lambda(\lambda m + \tilde{\zeta})\varphi(L) w(L)$$

$$L(w) = \int_0^L (f_2 + \lambda f_1) w\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi)\, d\xi\; w(L)$$
$$+(\lambda m + \tilde{\zeta}) f_1(L) w(L) + m f_4 w(L)$$

where $\zeta = (\pi)^{-1} \sin(\alpha\pi)\gamma$ and $\tilde{\zeta} = \zeta \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2 + \eta + \lambda}\, d\xi$.

Well-posedness

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $w \in H^1_L(0,L)$ problem (6) admits a unique solution $\varphi) \in H^1_L(0,L)$. Applying the classical elliptic regularity, it follows that $\varphi \in H^2(0,L)$. Therefore, the operator $\lambda I - A$ is surjective for any $\lambda > 0$. Consequently, using HilleñYosida theorem, we have the following results.

Theorem (Existence and uniqueness)

(1) *If $U_0 \in D(\mathcal{A})$, then system (1) has a unique strong solution*

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) *If $U_0 \in \mathcal{H}$, then system (1) has a unique weak solution*

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

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Lack of exponential stability

Theorem

The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.

Proof : We will examine two cases.
• **Case 1** $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(\sin x, 0, 0, 0)^T \in \mathcal{H}$, and denoting by $(\varphi, u, \phi, v)^T$ the image of $(\sin x, 0, 0, 0)^T$ by $\mathcal{A}^{-1}$, we see that $\phi(\xi) = |\xi|^{\frac{2\alpha-5}{2}} \sin L$. But, then $\phi \notin L^2(-\infty, +\infty)$, since $\alpha \in]0, 1[$. And so $(\varphi, u, \phi, v)^T \notin D(\mathcal{A})$.
• **Case 2** $\eta \neq 0$: We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the wave system (P) from being exponentially stable. Indeed We first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (\varphi, u, \phi, v)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda\varphi - u = 0, \\ \lambda u - \varphi_{xx} = 0, \\ \lambda\phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = 0, \\ \lambda v + \dfrac{1}{m}\varphi_x(L) + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0 \end{cases} \tag{7}$$

From $(7)_1 - (7)_2$ for such λ, we find

$$\lambda^2\varphi - \varphi_{xx} = 0. \tag{8}$$

Since $v = u(L)$, using $(7)_3$ and $(7)_4$, we get

$$\begin{cases} \varphi(0) = 0, \\ \left(\lambda + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \dfrac{\mu^2(\xi)}{\xi^2 + \lambda + \eta}\, d\xi\right) u(L) + \dfrac{1}{m}\varphi_x(L) \\ \qquad = \left(\lambda + \dfrac{\gamma}{m}(\lambda + \eta)^{\alpha - 1}\right)\lambda\varphi(L) + \dfrac{1}{m}\varphi_x(L) = 0. \end{cases} \tag{9}$$

The solution φ is given by

$$\varphi(x) = \sum_{i=1}^{2} c_i e^{t_i x}, \ t_1 = \lambda, \ t_2 = -\lambda. \tag{10}$$

Thus the boundary conditions may be written as the following system :

$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 \\ h(t_1)e^{t_1 L} & h(t_2)e^{t_2 L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

where we have set

$$h(r) = \frac{1}{m} r + \lambda^2 + \frac{\gamma}{m} \lambda (\lambda + \eta)^{\alpha - 1}.$$

Hence a non-trivial solution φ exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = detM(\lambda)$, thus the characteristic equation is $f(\lambda) = 0$.

Lemma

There exists $N \in \mathbb{N}$ *such that*

$$\{\lambda_k\}_{k\in\mathbf{Z}^*,|k|\geq N} \subset \sigma(\mathcal{A}) \tag{12}$$

where

$$\lambda_k = i\left(\frac{k\pi}{L} + \frac{1}{mk\pi}\right) + \frac{\tilde{\alpha}}{k^{3-\alpha}} + \frac{\beta}{|k|^{3-\alpha}} + o\left(\frac{1}{k^{3-\alpha}}\right), |k| \geq N, \tilde{\alpha} \in i\mathbb{R},$$

with

$$\beta = -\frac{\gamma}{m^2 L^{\alpha-2}\pi^{3-\alpha}}\cos(1-\alpha)\frac{\pi}{2}.$$

Moreover for all $|k| \geq N$*, the eigenvalues* λ_k *are simple.*

The operator $\mathcal{A}$ has a non exponential decaying branch of eigenvalues.

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Asymptotic behavior

Theorem (Borichev-Tomilov)

Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup on a Hilbert space. If

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \sup_{|\beta|\geq 1} \frac{1}{\beta^l} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}\mathcal{H}} < M$$

for some l, then there exist c such that

$$\|e^{\mathcal{A}t} U_0\| \leq \frac{c}{t^{\frac{1}{l}}} \|U_0\|_{D(\mathcal{A})}$$

Theorem (Arendt-Batty)

Let $\mathcal{A}$ be the generator of a uniformly bounded C_0. semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space H. If :

(i) *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

(ii) *The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set,*

then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e,
$\|S(t)x\| \to 0$ as $t \to \infty$ for any $x \in H$

Lemma

We have

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ if } \eta \neq 0,$$
$$i\mathbb{R}^* \subset \rho(\mathcal{A}) \text{ if } \eta = 0$$

where $\mathbb{R}^* = \mathbb{R} - \{0\}$.

Proof

Let $\lambda \in \mathbb{R}$. Let $F = (f_1, f_2, f_3, f_3)^T \in \mathcal{H}$ be given, and let $X = (\varphi, u, \phi, v)^T \in D(\mathcal{A})$ be such that

$$(i\lambda I - \mathcal{A})X = F. \tag{13}$$

Equivalently, we have

$$\begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u - \varphi_{xx} = f_2, \\ i\lambda\phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = f_3, \\ i\lambda v + \frac{1}{m}\varphi_x(L) + \frac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = f_4, \end{cases} \tag{14}$$

From $(14)_1$ and $(14)_2$, we have

$$\lambda^2\varphi + \varphi_{xx} = -(f_2 + i\lambda f_1)$$

with $\varphi(0) = 0$. Suppose that $\lambda \neq 0$. Then

$$\varphi(x) = c_1 \sin\lambda x - \frac{1}{\lambda}\int_0^x (f_2(\sigma) + i\lambda f_1(\sigma)) \sin\lambda(x-\sigma)\, d\sigma, \tag{15}$$

$$\varphi_x(x) = c_1\lambda \cos\lambda x - \int_0^x (f_2(\sigma) + i\lambda f_1(\sigma)) \cos\lambda(x-\sigma)\, d\sigma. \tag{16}$$

From $(14)_3$ and $(14)_4$, we have

$$\phi(\xi) = \frac{u(L)\mu(\xi) + f_3(\xi)}{i\lambda + \xi^2 + \eta}$$

$$\left(i\lambda + \frac{\zeta}{m}\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi\right) u(L) + \frac{1}{m}\varphi_x(L) + \frac{\zeta}{m}\int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi = f_4. \tag{17}$$

Since

$$\frac{\zeta}{m}\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{i\lambda+\xi^2+\eta}\,d\xi=\frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1}$$

and

$$u(L)=i\lambda\varphi(L)-f_1(L),$$

using (15), (16) and (17), we get

$$\begin{aligned}\lambda c_1\left[iI\sin\lambda L+\frac{1}{m}\cos\lambda L\right]&=J+If_1(L)+iI\int_0^L(f_2(\sigma)+i\lambda f_1(\sigma))\sin\lambda(L-\sigma)\,d\sigma\\&+\frac{1}{m}\int_0^L(f_2(\sigma)+i\lambda f_1(\sigma))\cos\lambda(L-\sigma)\,d\sigma\end{aligned} \tag{18}$$

where

$$\begin{aligned}I&=i\lambda+\frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1},\\J&=f_4-\tfrac{\zeta}{m}\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi.\end{aligned}$$

We set

$$\begin{aligned} g(\lambda) &= iI \sin \lambda L + \frac{1}{m} \cos \lambda L \\ &= -\lambda \sin \lambda L + \frac{1}{m} \cos \lambda L + i\frac{\gamma}{m}(i\lambda + \eta)^{\alpha-1} \sin \lambda L \\ &= -\lambda \sin \lambda L + \frac{1}{m} \cos \lambda L + \frac{\gamma}{m}(\lambda^2 + \eta^2)^{\frac{\alpha-1}{2}} \sin(1-\alpha)\theta \sin \lambda L \\ &\quad + i\frac{\gamma}{m}(\lambda^2 + \eta^2)^{\frac{\alpha-1}{2}} \cos(1-\alpha)\theta \sin \lambda L \end{aligned}$$

where $\theta \in]-\pi/2, \pi/2[$ such that

$$\begin{aligned} \cos\theta &= \frac{\eta}{\sqrt{\lambda^2+\eta^2}} \\ \sin\theta &= \frac{\lambda}{\sqrt{\lambda^2+\eta^2}} \end{aligned}$$

It is clear that

$$g(\lambda) \neq 0 \quad \forall \lambda \in \mathbb{R}.$$

Hence $i\lambda - \mathcal{A}$ is surjective for all $\lambda \in \mathbb{R}^*$.

Now, if $\lambda = 0$ and $\eta \neq 0$, the system (14) is reduced to the following system

$$\begin{cases} u = -f_1, \\ \varphi_{xx} = -f_2, \\ (\xi^2 + \eta)\phi - u(L)\mu(\xi) = f_3, \\ \frac{1}{m}\varphi_x(L) + \frac{\zeta}{m}\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = f_4. \end{cases} \tag{19}$$

We deduce from $(19)_2$

$$\varphi(x) = -\int_0^x \int_0^s f_2(r)\, dr\, ds + Cx.$$

From $(19)_1, (19)_3$ and $(19)_4$, we have

$$-\frac{\gamma}{m}\eta^{\alpha-1} f_1(L) + \frac{1}{m}\varphi_x(L) = f_4 - \frac{\zeta}{m}\int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{\xi^2 + \eta}\, d\xi.$$

We find

$$C = \int_0^L f_2(r)\, dr + \gamma\eta^{\alpha-1} f_1(L) + m f_4 - \zeta\int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{\xi^2 + \eta}\, d\xi.$$

Hence $\mathcal{A}$ is surjective.

Lemma

Let $\mathcal{A}^*$ be the adjoint operator of $\mathcal{A}$. Then

$$\mathcal{A}^* \begin{pmatrix} \varphi \\ u \\ \phi \\ v \end{pmatrix} = \begin{pmatrix} -u \\ -\varphi_{xx} \\ -(\xi^2+\eta)\phi - u(L)\mu(\xi) \\ \frac{1}{m}\varphi_x(L) + \frac{\zeta}{m}\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\,d\xi \end{pmatrix} \qquad (20)$$

with domain

$$D(\mathcal{A}^*) = \left\{ \begin{array}{l} (\varphi,u,\phi,v)^T \text{ in } \mathcal{H} : \varphi \in H^2(0,L) \cap H_L^1(0,L), u \in H_L^1(0,L), v \in \mathbb{C} \\ -(\xi^2+\eta)\phi - u(L)\mu(\xi) \in L^2(-\infty,+\infty), u(L) = v \\ |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\} \qquad (21)$$

Theorem

$\sigma_r(\mathcal{A}) = \emptyset$, where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$.

Proof

Since $\lambda \in \sigma_r(\mathcal{A}), \overline{\lambda} \in \sigma_p(\mathcal{A}^*)$ the proof will be accomplished if we can show that $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$. This is because obviously the eigenvalues of A are symmetric on the real axis.

Theorem

The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and

$$\|S_{\mathcal{A}}(t)U_0\|_{\mathcal{H}} \leq \frac{1}{t^{1/(4-2\alpha)}}\|U_0\|_{D(\mathcal{A})}.$$

Proof
We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u - \varphi_{xx} = f_2, \\ i\lambda\phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = f_3, \\ i\lambda v + \frac{1}{m}\varphi_x(L) + \frac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = f_4, \end{cases} \tag{22}$$

where $F = (f_1, f_2, f_3, f_4)^T$. Taking inner product in $\mathcal{H}$ with U and using (4) we get

$$|Re\langle \mathcal{A}U, U\rangle| \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{23}$$

This implies that

$$\zeta \int_{-\infty}^{+\infty} (\xi^2+\eta)(\varphi_i(\xi,t))^2\, d\xi \le \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{24}$$

and, applying $(22)_1$, we obtain

$$||\lambda||\varphi(L)| - |f_1(L)||^2 \le |u(L)|^2.$$

We deduce that

$$|\lambda|^2|\varphi(L)|^2 \le c|f_1(L)|^2 + c|u(L)|^2.$$

Moreover, from $(22)_4$, we have

$$\varphi_x(L) = -im\lambda u(L) - \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi + mf_4.$$

Then

$$\begin{aligned}
|\varphi_x(L)|^2 &\le 2m^2|\lambda|^2|u(L)|^2 + 2m^2f_4^2 + 2\zeta^2\left|\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\, d\xi\right|^2 \\
&\le 2m^2|\lambda|^2|u(L)|^2 + 2m^2f_4^2 + 2\zeta^2\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)^{-1}|\mu(\xi)|^2\, d\xi\right)\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi
\\
&\le 2m^2|\lambda|^2|u(L)|^2 + c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c'\|F\|_{\mathcal{H}}^2.
\end{aligned} \tag{25}$$

From $(22)_3$, we obtain

$$u(L)\mu(\xi) = (i\lambda + \xi^2 + \eta)\phi - f_3(\xi). \tag{26}$$

By multiplying $(26)_1$ by $(i\lambda + \xi^2 + \eta)^{-1}\mu(\xi)$, we get

$$(i\lambda + \xi^2 + \eta)^{-1}u(L)\mu^2(\xi) = \mu(\xi)\phi - (i\lambda + \xi^2 + \eta)^{-1}\mu(\xi)f_3(\xi). \tag{27}$$

Hence, by taking absolute values of both sides of (27), integrating over the interval $]-\infty, +\infty[$ with respect to the variable ξ and applying Cauchy-Schwartz inequality, we obtain

$$S|u(L)| \leq U\left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi|^2 \, d\xi\right)^{\frac{1}{2}} + V\left(\int_{-\infty}^{+\infty} |f_3(\xi)|^2 \, d\xi\right)^{\frac{1}{2}} \tag{28}$$

where

$$S = \int_{-\infty}^{+\infty} (|\lambda| + \xi^2 + \eta)^{-1} |\mu(\xi)|^2 \, d\xi$$

$$U = \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)^{-1} |\mu(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

$$V = \left(\int_{-\infty}^{+\infty} (|\lambda| + \xi^2 + \eta)^{-2} |\mu(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$

Thus, by using again the inequality $2PQ \leq P^2 + Q^2, P \geq 0, Q \geq 0$, we get

$$S^2 |u(L)|^2 \leq 2U^2 \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta) |\phi|^2 \, d\xi \right) + 2V^2 \left(\int_{-\infty}^{+\infty} |f_3(\xi)|^2 \, d\xi \right). \tag{29}$$

We deduce that

$$|u(L)|^2 \leq c |\lambda|^{2-2\alpha} \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + c \|F\|_{\mathcal{H}}^2. \tag{30}$$

Let us introduce the following notation

$$\mathcal{I}_\varphi(\alpha) = |u(\alpha)|^2 + |\varphi_x(\alpha)|^2$$

$$\mathcal{E}_\varphi(L) = \int_0^L q(x)\mathcal{I}_\varphi(s)\,ds.$$

Lemma

Let $q \in H^1(0,L)$. We have that

$$\mathcal{E}_\varphi(L) = [q\mathcal{I}_\varphi]_0^L + R \tag{31}$$

where R satisfies

$$|R| \leq C\mathcal{E}_\varphi(L) + \|q^{1/2}F\|^2_{\mathcal{H}}.$$

for a positive constant C.

Proof

To get (31), let us multiply the equation $(22)_2$ by $q\overline{\varphi}_x$ Integrating on $(0,L)$ we obtain

$$i\lambda\int_0^L uq\overline{\varphi}_x\,dx-\int_0^L \varphi_{xx}q\overline{\varphi}_x\,dx=\int_0^L f_2q\overline{\varphi}_x\,dx$$

or

$$-\int_0^L uq(\overline{i\lambda\varphi_x})\,dx-\int_0^L q\varphi_{xx}\overline{\varphi}_x\,dx=\int_0^L f_2q\overline{\varphi}_x\,dx.$$

Since $i\lambda\varphi_x=u_x+f_{1x}$ taking the real part in the above equality results in

$$-\frac{1}{2}\int_0^L q\frac{d}{dx}|u|^2\,dx-\frac{1}{2}\int_0^L q\frac{d}{dx}|\varphi_x|^2\,dx=Re\int_0^L f_2q\overline{\varphi}_x\,dx+Re\int_0^L uq\overline{f}_{1x}\,dx.$$

Performing an integration by parts we get

$$\int_0^L q'(s)[|u(s)|^2+|\varphi_x(s)|^2]\,ds=[q\mathcal{I}_\varphi]_0^L+R$$

where

$$R = 2Re\int_0^L f_2 q\overline{\varphi}_x\,dx + 2Re\int_0^L uq\overline{f}_{1x}\,dx.$$

If we take $q(x) = \int_0^x e^{ns}\,ds = \frac{e^{nx}-1}{n}$ (Here n will be chosen large enough) in Lemma 5.3 we arrive at

$$\mathcal{E}_\varphi(L) = q(L)\mathcal{I}_\varphi(L) + R. \tag{32}$$

Also, we have

$$\begin{aligned} |R| &\leq \int_0^L q(x)(|u(s)|^2 + |\varphi_x(s)|^2)\,ds + \int_0^L q(x)(|f_2(s)|^2 + |f_{1x}(s)|^2)\,ds \\ &\leq C\frac{e^{Ln}}{n}\|F\|^2_{\mathcal{H}} + \frac{c'}{n}\mathcal{E}_\varphi(L) \end{aligned} \tag{33}$$

Using inequalities (32) and (33) we conclude that there exists a positive constant C such that

$$\int_0^L \mathcal{I}_\varphi(s)\,ds \leq C\mathcal{I}_\varphi(L) + C'\|F\|^2_{\mathcal{H}}. \tag{34}$$

provided n is large enough.

Since that

$$\int_{-\infty}^{+\infty} (\phi(\xi))^2 \, d\xi \leq C \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi(\xi))^2 \, d\xi \leq C \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}.$$

Substitution of inequalities (25)and (30) into (34) we get that

$$\|U\|_{\mathcal{H}}^2 \leq C(|\lambda|^{4-2\alpha} + |\lambda|^{2-2\alpha} + 1)\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C'(|\lambda|^2 + 1)\|F\|_{\mathcal{H}}^2.$$

So we have

$$\|U\|_{\mathcal{H}} \leq C|\lambda|^{4-2\alpha}\|F\|_{\mathcal{H}}.$$

The conclusion then follows by applying the Theorem 4.

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