Regular propagators of bilinear quantum systems

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Abstract framework

In a separable Hilbert space (possibly infinite dimensional) $\mathcal{H}$, we consider the controllability problem of the abstract Schrödinger equation:

$$\partial_t \psi = A\psi + u(t)B\psi$$

where

- A is a skew-adjoint operator,
- B a control potential and
- u is the control command which is a real valued function.

Abstract framework

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$$\partial_t \psi = A\psi + u(t)B\psi$$

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For instance, if $\boldsymbol{A}$ is skew-adjoint and $\boldsymbol{B}$ is $\boldsymbol{A}$-bounded with $\boldsymbol{A}$-bound zero then if $\boldsymbol{u}$ is piecewise constant $\boldsymbol{u} = \sum_{j=1}^{p} u_j \mathbf{1}_{[\tau_j,\tau_{j+1})}$ where $u_1, \dots, u_p \in \mathbf{R}$ by concatenation we obtain a solution: $s = \tau_1 < \tau_2 < \dots < \tau_{p+1}$ on pose

$$x(t,s) = e^{(t-\tau_j)(A+u_jB)} \circ e^{(\tau_j-\tau_{j-1})(A+u_{j-1}B)} \circ \dots \circ e^{(\tau_2-\tau_1)(A+u_1B)} x_s,$$

for t in (τ_j, τ_{j+1}).

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In the sequel, we denote by $\boldsymbol{\Upsilon^u_{t,s}}$ the maps defined by $\boldsymbol{x(t,s) =: \Upsilon^u_{t,s} x_s}$ and we will consider the attainable set:

$$\bigcup_{T>s} \left\{ \Upsilon^u_{T,s} x_s, u \in Z(0,T) \right\}$$

for some $\boldsymbol{x_s}$ (and even $\boldsymbol{s = 0}$) and a class of control $\boldsymbol{Z(0,T)}$ to be chosen.

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We want to give bounds on the attainable set.

An example

Our main example is the Schrödinger equation

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\frac{1}{2}\Delta\psi + V(x)\psi(x,t) + u(t)W(x)\psi(x,t),$$

where Δ is the Laplace-Beltrami operator on Ω, a riemannian manifold, and $V : \Omega \to \mathbb{R}$ is a potential.

The system is submitted to an excitation by an external field (*e.g.* a laser) $W : \Omega \to \mathbb{R}$ and u is a real function of the time modulating its amplitude.

In our abstract framework, we have

$$A = \frac{\mathrm{i}}{2}\Delta - \mathrm{i}V(x) \quad \text{and} \quad B = -\mathrm{i}W(x).$$

Some bibliography

J. M. Ball, J. E. Marsden, et M. Slemrod.
Controllability for distributed bilinear systems.
SIAM J. Control Optim., 20(4):575–597, 1982.

Theorem ([Ball, Marsden & Slemrod 82, Theorem 3.6])

Let X be a Banach space of infinite dimension, A the generator of a C^0 semi-group on X, and B a bounded operator on X. Then for all $T \geq 0$, the input-output mapping $u \mapsto \Upsilon^u_{T,0} x_0$ has a unique continuous extension to $L^1([0,T])$ and the attainable set

$$\bigcup_{r>1} \bigcup_{T \geq 0} \bigcup_{u \in L^r([0,T],\mathbf{R})} \left\{ \Upsilon^u_{t,0} x_0, t \in [0,T] \right\}$$

is included in a countable union of compact subsets of X.

Some bibliography

K. Beauchard.
Local controllability of a 1-D Schrödinger equation. J. Math. Pures Appl., 84(7):851–956, 2005.

Some bibliography

K. Beauchard et C. Laurent.
Local controllability of 1D linear et nonlinear Schrödinger equations with bilinear control. J. Math. Pures Appl., 94(5):520–554, 2010.

Some bibliography

Theorem ([Beauchard & Laurent 10])

Let $T > 0$ *and* $\mu \in H^3((0,1),\mathbb{R})$ *such that* $\frac{c}{k^3} \leqslant |\langle \mu\varphi_1, \varphi_k\rangle|$, *for all* $k \in \mathbb{N}^*$. *There exist* $\delta > 0$ *and a* C^1 *mapping* $\Gamma : \mathcal{V}_T \to L^2((0,T),\mathbb{R})$ *where*

$$\mathcal{V}_T := \{\psi_f \in H^3_{(0)}((0,1)), \|\psi_f\| = 1, \|\psi_f - \psi_1(T)\|_{H^3} < \delta\},$$

such that $\Gamma(\psi_1(T)) = 0$ *and for all* $\psi_f \in \mathcal{V}_T$, *is the solution of*

$$\begin{cases} i\frac{\partial\psi}{\partial t}(t,x) = -\frac{\partial^2\psi}{\partial x^2}(t,x) - u(t)\mu(x)\psi(t,x), x \in (0,1), t \in (0,T), \\ \psi(t,0) = \psi(t,1) = 0, \end{cases}$$

with initial condition $\phi(0) = \varphi_1$ *and* $u = \Gamma(\psi_f)$ *satisfies* $\psi(T) = \psi_f$.

$H^s_{(0)}((0,1))$ the domain of $|A|^{s/2}$ where A is the Laplace-Dirichlet operator on $(0,1)$ and φ_k is "the" k-the normalised eigenvector.

Propagator on a Hilbert space

Let I be a real interval and

$$\Delta_I := \{(s,t) \in I^2 \mid s \leq t\}.$$

A family $(s,t) \in \Delta_I \mapsto X(s,t)$ of linear (contractions), on a Hilbert space $\mathcal{H}$, strongly continuous in t and s and such that

1. for any $s < r < t$, $X(t,s) = X(t,r)X(r,s)$,
2. $X(t,t) = I_{\mathcal{H}}$,

is called a (contraction) propagator on $\mathcal{H}$.

Bounded variation functions

A family $t \in I \mapsto U(t) \in E$, E a subset of a Banach space X, is in $BV(I,E)$ if there exists $N \geq 0$ such that

$$\sum_{j=1}^{n} \|U(t_j) - U(t_{j-1})\|_X \leq N$$

for any partition $a = t_0 < t_1 < \ldots < t_n = b$ of the interval (a,b).

The mapping

$$U \in BV(I,E) \mapsto \sup_{a=t_0<t_1<\ldots<t_n=b} \sum_{j=1}^{n} \|U(t_j) - U(t_{j-1})\|_X$$

is a semi-norm on $BV(I,E)$ denoted $\|\cdot\|_{BV(I,E)}$ and called total variation.

Our assumptions

Let us now fix some scalar function $u : I \mapsto \mathbb{R}$ and define

$$A(t) = A + u(t)B.$$

Let I be a real interval and $\mathcal{D}$ dense subset of $\mathcal{H}$

1. $A(t)$ is a maximal dissipative operator on $\mathcal{H}$ with domain $\mathcal{D}$,
2. $t \mapsto A(t)$ has bounded variation from I to $L(\mathcal{D}, \mathcal{H})$, where $\mathcal{D}$ is endowed with the graph topology associated to $A(a)$ for $a = \inf I$,
3. $M := \sup_{t \in I} \left\| (1 - A(t))^{-1} \right\|_{L(\mathcal{H}, \mathcal{D})} < \infty$.

The map $t \mapsto A(t)$ is not necessarily continuous but admits right and left limits which are equal except on a at most countable set.

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A theorem by T. Kato

The the core of our analysis is the following result due to Tosio Kato (1953).

Theorem

If $t \in I \mapsto A(t)$ satisfies the above assumptions, then there exists a unique (contraction) propagator $X : \Delta_I \to L(\mathcal{H})$ such that if $\psi_0 \in \mathcal{D}$ then $X(t,s)\psi_0 \in \mathcal{D}$ and for $(t,s) \in \Delta_I$

$$\|A(t)X(t,s)\psi_0\| \leq M e^{M\|A\|_{BV(I,L(\mathcal{D},\mathcal{H}))}} \|A(s)\psi_0\|$$

and in this case $X(t,s)\psi_0$ is strongly left differentiable in t and right differentiable in s with derivative (when $t = s$) $A(t+0)\psi_0$ and $-A(t-0)\psi_0$ respectively.

An immediate consequence

In the estimates

$$\|A(t)X(t,s)\psi_0\| \leq Me^{M\|A\|_{BV(I,L(\mathcal{D},\mathcal{H}))}}\|A(s)\psi_0\|$$

we "read" that the attainable set from any eigenvector is included in the domain.

When $u(t)$ is a constant u then these estimates can be extend to any (semi-)norm of the type

$$\psi \in D((A+uB)^k) \mapsto \|(A+uB)^k\psi\|.$$

First question

We consider the attainable set from an eigenvector. Since eigenvectors are in the domain of $\boldsymbol{A^k}$ for any integer $\boldsymbol{k}$ then the first question we considered was:

At what extent is these still true for the non-autonomous case ?

The weak coupling

Let k be a non negative real. A couple of skew-adjoint operators (A, B) is k-weakly coupled if

1. A is invertible with bounded inverse from $D(A)$ to $\mathcal{H}$,
2. there exist $c \geq 0$ and $c' \geq 0$ such that $B - c$ and $-B - c'$ generate contraction semigroups on $D(|A|^{k/2})$ for the norm $\psi \mapsto \||A|^{k/2} u\|$.

We set, for every positive real k,

$$\|\psi\|_{k/2} = \sqrt{\langle |A|^k \psi, \psi\rangle}.$$

The optimal exponential growth is defined by

$$c_k(A, B) := \sup_{t \in \mathbb{R}} \frac{\log \|e^{tB}\|_{L(D(|A|^{k/2}), D(|A|^{k/2}))}}{|t|}.$$

If B is A-bounded, we set:

$$\|B\|_A := \inf_{\lambda>0} \|B(\lambda - A)^{-1}\|.$$

Theorem

For any $u \in BV([0,T],\mathbb{R}) \cap B_{L^\infty([0,T])}(0, 1/\|B\|_A)$, *there exists a family of contraction propagators in* $\mathcal{H}$ *that extends uniquely as propagators to* $D(|A|^{k/2})$*:* $\Upsilon^u : \Delta_{[0,T]} \to L(D(|A|^{k/2}))$ *such that for any* $t \in [0,T]$, *for any* $\psi_0 \in D(|A|^{k/2})$

$$\|\Upsilon^u_t(\psi_0)\|_{k/2} \leq e^{c_k(A,B)\left|\int_0^t u\right|}\|\psi_0\|_{k/2}.$$

Moreover, there exists m *(depending only on* A, B *and* $\|u\|_{L^\infty([0,T])}$*)*

$$\|\Upsilon^u_t(\psi_0)\|_{1+k/2} \leq m e^{m\|u\|_{BV([0,T],\mathbb{R})}} e^{c_k(A,B)\left|\int_0^t u\right|}\|\psi_0\|_{1+k/2}$$

where Υ^u_t *stands for the propagator* $X(t,0)$ *associated with* $A + u(t)B$.

A compactness result

Theorem (Helly's selection theorem)

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $BV(I,\mathbb{R})$, where I is a compact interval. If

1. *there exists $M>0$ such that for all $n\in\mathbb{N}$, $\|f_n\|_{BV(I,\mathbb{R})}\leq M$,*
2. *there exists $x_0\in I$ such that $(f_n(x_0))_{n\in\mathbb{N}}$ is bounded.*

Then $(f_n)_{n\in\mathbb{N}}$ admits a pointwise convergent subsequence.

For every ψ_0 in $D(|A|^{k/2})$, the end-point mapping

$$\begin{aligned}\Upsilon(\psi_0) : BV([0,T],K)\cap L^1([0,T]) &\to D(|A|^{k/2})\\ u &\mapsto \Upsilon^u_T(\psi_0)\end{aligned}$$

is continuous for the topology corresponding to the previous lemma.

An upper bound

Theorem

Let $\psi_0 \in D(|A|^{k/2})$. *Then*

$$\bigcup_{L,T,a>0} \left\{\alpha \Upsilon_t^u(\psi_0), \|u\|_{BV([0,T],\mathbb{R})\cap L^1([0,T])} \leq L, t \in [0,T], |\alpha| \leq a\right\}$$

is a meagre set (in the sense of Baire) in $L^\infty(I, D(|A|^{k/2}))$ *as a union of relatively compact subsets.*

Galerkin Approximation

For a Hilbert basis $\Phi = (\phi_k)_{k\in\mathbb{N}}$ of $\mathcal{H}$ made of eigenvectors of A, let

$$\pi_N^\Phi : \psi \in \mathcal{H} \mapsto \sum_{j\leq N} \langle \phi_j, \psi\rangle \phi_j \in \mathcal{H}$$

be the orthogonal projector to $\mathcal{L}_N^\Phi = \mathrm{span}(\phi_1, \ldots, \phi_N)$.
The Galerkin approximation of order N of our system is the system

$$\dot{x} = (A^{(\Phi,N)} + uB^{(\Phi,N)})x$$

where

$$A^{(\Phi,N)} := A^{(\Phi,N)} = \pi_N^\Phi A_{\restriction \mathcal{L}_N^\Phi} \quad \text{and} \quad B^{(\Phi,N)} := B^{(\Phi,N)} = \pi_N^\Phi B_{\restriction \mathcal{L}_N^\Phi}.$$

$(A^{(\Phi,N)}, B^{(\Phi,N)})$ satisfies the same assumptions as (A, B). We can define the associated contraction propagator $X^u_{(\Phi,N)}(t, 0)$.

Good Galerkin Approximation

Theorem

If $B(1-A)^{-1}$ *compact then for* s *real with* $0 \leq s < k$ *for every* $\varepsilon > 0$, $L \geq 0$, $n \in \mathbb{N}$, *and* $(\psi_j)_{1 \leq j \leq n}$ *in* $D(|A|^{k/2})^n$ *there exists* $N \in \mathbb{N}$ *such that for any* $u \in \mathcal{R}((0,T])$,

$$\|u\|_{BV([0,T])} < L \Rightarrow \|\Upsilon_t^u(\psi_j) - X_{(N)}^u(t,0)\pi_N\psi_j\|_{s/2} < \varepsilon,$$

for every $t \geq 0$ *and* $j = 1, \ldots, n$.

Hence if these finite dimensional systems are controllable with a uniform bound on the total variation of the needed controls u then the system is approximately controllable.

Second question

Is it possible to consider a larger class of controls?

Extension to Radon measures

We denote by $\mathcal{R}(I)$ the set of signed Radon measures on I.

- A positive Radon measure is a locally finite and inner regular borelian measure.
- A signed Radon measure μ can be written as the difference $\mu = \mu^+ - \mu^-$ (Hahn-Jordan decomposition) of two positive Radon measures μ^+ and μ^- (with disjoint supports).
- Denote $|\mu| = \mu^+ + \mu^-$ and $|\mu|(I)$ the total variation of μ on I.

Extension to Radon measures

We denote by $\mathcal{R}(I)$ the set of signed Radon measures on I.

- The cumulative function of a Radon measure has local bounded variation, The total variations of the cumulative function and the measure coincide.
- We consider that $(u_n)_{n\in\mathbb{N}} \in \mathcal{R}(I)$ converges to $u \in \mathcal{R}(I)$ if:
  - $\sup_n |u_n|(I) < +\infty$ (bounded total variations);
  - $u_n((0,t]) \to u((0,t])$ for all $t \in I$.

Extension to Radon measures

We denote by $\mathcal{R}(I)$ the set of signed Radon measures on I.

- Any $L^1_{\text{loc}}(I)$ can be considered as a density of an absolute continuous function and thus as a Radon measure .

Extension to Radon measures

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Assumption

The couple (A, B), with A generator of a contraction semi-group on $\mathcal{H}$ is such that

1. *there exist $c \geq 0$ and $c' \geq 0$ such that $B - c$ and $-B - c'$ generate contractions semi-groups on $\mathcal{H}$ leaving $D(A)$ invariant,*
2. *for all $u \in \mathcal{R}((0, T])$,*

$$t \in [0, T] \mapsto \mathcal{A}(t) := e^{u((0,t])B} A e^{-u((0,t])B}$$

is a family of contraction semi-group generators with domain $D(A)$ and:

- $\mathcal{A}$ *has bounded variation from $[0, T]$ to $L(D(A), \mathcal{H})$,*
- $\sup_{t \in [0,T]} \left\| (1 - \mathcal{A}(t))^{-1} \right\|_{L(\mathcal{H}, D(A))} < +\infty$.

Lemma (Continuity in the control)

Let $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{R}(I)$ be uniformly bounded (for the total variation on I) such that the distributions functions are almost everywhere convergent to the distribution function of some $v \in \mathcal{R}(I)$. Let

$$\mathcal{A}_n(t) = e^{-v_n((0,t])B} A e^{v_n(0,t])B}$$

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and X_n (resp. X) the propagators associated with $\mathcal{A}_n$ (resp. $\mathcal{A}$). If $\sup_{n\in\mathbb{N}} \|\mathcal{A}_n\|_{BV(I,L(D(A),\mathcal{H}))} < +\infty$, then $X_n(t,s)$ converges strongly to $X(t,s)$ locally uniformly in $s,t \in I$ (with $s \le t$).

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If $\sup_{n\in\mathbb{N}} \|\mathcal{A}_n\|_{BV(I,L(D(A),\mathcal{H}))} < +\infty$, *then* $X_n(t,s)$ *converges strongly to* $X(t,s)$ *locally uniformly in* $s, t \in I$ *(with* $s \leq t$*).*

Assumption

1. A *generates a contraction semi-group on* $\mathcal{H}$ *with domain* $D(A)$,
2. *there exist* $c \geq 0$ *and* $c' \geq 0$ *such that* $B - c$ *and* $-B - c'$ *generate contraction semi-groups on* $\mathcal{H}$ *leaving* $D(A)$ *invariant,*
3. $t \in \mathbb{R} \mapsto e^{tB} A e^{-tB} \in L(D(A), \mathcal{H})$ *is locally Lipschitz.*

Assumption

The couple (A, B) *with* A *generator of a contraction semi-group on* $\mathcal{H}$ *and* B *an operator on* $\mathcal{H}$ *such that* $D(A) \subset D(B)$ *and* $A + uB$ *generates a contraction semi-group on* $\mathcal{H}$ *for any* $u \in \mathbf{R}$*, with domain* $D(A)$*.*

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Proposition

Let $t\mapsto Y_t^u$ *be the propagator (with* $s=0$*) associated with*

$$\mathcal{A}(t):=e^{-u((0,t])B}Ae^{u((0,t])B},$$

for $u\in\mathcal{R}((0,T])$, *and* Υ_t^u *the one (with* $s=0$*) associated with* $A+u(t)B$ *for* $u\in BV([0,T],\mathrm{R})$.
Then for any $\psi_0\in\mathcal{H}, t\in[0,T]$ *the mapping*

$$\Upsilon_t(\psi_0): u\mapsto \Upsilon_t^u(\psi_0)\in\mathcal{H}$$

ha a unique continuous extension to $\mathcal{R}((0,T])$, *denoted* $\Upsilon_t(\psi_0)$, *satisfying*

$$\Upsilon_t^u(\psi_0)=e^{u((0,t])B}Y_t^u(\psi_0), \forall u\in\mathcal{R}((0,T]), \forall t\in[0,T].$$

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Proposition

Let $T > 0$ and $\psi_0 \in \mathcal{H}$. The for any $L > 0$, the set

$$\{\Upsilon_t^u(\psi_0) \ : \ u \in \mathcal{R}((0,T]), |u|((0,T]) \leq L, t \in [0,T]\}$$

is relatively compact in $\mathcal{H}$.

So

$$\bigcup_{L,T>0} \left\{\Upsilon_t^u(\psi_0), u \in \mathcal{R}((0,T]), |u|((0,T]) \leq L, t \in [0,T]\right\}$$

is included in a countable union of compact subsets of $\mathcal{H}$.

The case of bounded controls potentials B

The construction of a solution when B is bounded on a Banach space X can be done by a Dyson expansion (iterations Duhamel's formula)

$$\Upsilon^u_{t,s}\psi_0 = e^{(t-s)A}\psi_0 + \sum_{n=1}^{\infty} \int_{s<s_1<s_2<\ldots<s_n\leq t} e^{(t-s_n)A} B e^{(s_n-s_{n-1})A} \circ \cdots$$
$$\cdots \circ B e^{(s_2-s_1)A} B e^{(s_1-s)A}\psi_0 \, \mathrm{du}(s_1)\ldots\mathrm{du}(s_n)$$

without any assumption but the one needed to make the sum and the integrals convergent.

The case of bounded controls potentials $\boldsymbol{B}$

The construction of a solution when $\boldsymbol{B}$ is bounded on a Banach space $\boldsymbol{X}$ can be done by a Dyson expansion (iterations Duhamel's formula)

The results on the continuity are still valid and Helly's selection theorem can be used as well.

The case of bounded controls potentials B

The construction of a solution when B is bounded on a Banach space X can be done by a Dyson expansion (iterations Duhamel's formula)

The results on the continuity are still valid and Helly's selection theorem can be used as well.

Proposition

For any $T > 0$, there exists a unique "continuous" extension to $\mathcal{R}([0, T])$ of the input-output

$$u \mapsto \Upsilon^u_{T,0} \in L(X, X)$$

and for all $\psi_0 \in \mathcal{H}$,

$$\bigcup_{T \geq 0} \bigcup_{u \in L^1([0,T],\mathrm{R})} \{\Upsilon^u_{t,0}\psi_0, t \in [0, T]\}$$

is included in a countable union of compacts subsets of $\mathcal{X}$.