Stability in the identification of a scalar potential by a partial elliptic DtN map

Mourad Choulli

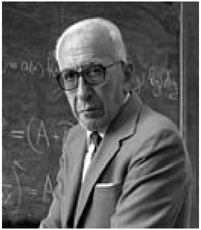
Institut Élie Cartan de Lorraine

Stability of non-conservative systems,Valenciennes, July 4th-7th, 2016

Summary

Summary

Alberto Pedro Calderón



Calderón, On an inverse boundary problem, *Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matemática, Rio de Janeíro* (1980), 65-73.

By A.P. Calderon

In this note we shall discuss the following problem. Let D be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with Lipschitzian boundary ∂D, and γ be a real bounded measurable function in D with a positive bower bound. Consider the differential operator

$$L_\gamma(W) = \nabla.(\gamma\nabla W)$$

acting on functions of $H^1(D)$ and the quadratic form $Q_\gamma(\phi)$, where the functions ϕ are restrictions to ∂D of functions in $H^1(\mathbb{R}^n)$, defined by

$$Q_\gamma(\phi) = \int_D \gamma(\nabla W)^2 dx \quad , W \in H^1(\mathbb{R}^n),\ W\Big|_{\partial D} = \phi$$

$$L_\gamma W = \nabla.(\gamma\nabla W) = 0 \quad \text{in } D.$$

The problem is then to decide whether γ is uniquely determined by Q_γ and to calculate γ in terms Q_γ, if γ is indeed determined by Q_γ.

This problems originates in the following problem of electrical prospection. If D represents an in-homogeneous conducting body with electrical conductivity γ, determine γ by means of direct current steady state electrical measurements carried aut on the surface of D, that is, without penetrating D. In this physicak situation $Q_\gamma(\phi)$ represents the power necessary to maintain an electrical potential γ on ∂D.

In principle Q_γ can be determined through measurements effected on ∂D and contains all the information about γ which can be thus obtained.

But let us retwen to our mathematical problem. Let us introduce the following norms in the space of functions γ on ∂D and in the space of quadratic forms $Q(\phi)$

$$\|\phi\|^2 = \int_D |\nabla u|^2 dx \quad ; u\Big|_{\partial D} = \phi \ , \ \Delta u = 0 \quad \text{in } D.$$

$$\|Q\| = \sup_{\|\phi\| \leq 1} |Q(\phi)|$$

Then the mapping

$$\Phi : \gamma \to Q_\gamma.$$

is bounded and analytic in the subset of $L^\infty(D)$ consisting of functions which are real and have a positive lower bound. Our goal is then to determine whether Φ is inyective, and invert Φ if this is the case. This we are yet unable to do, and is, as far as we known, an open problem. However we shall show that $d\Phi|_{\gamma = \text{const.}}$ is indeed inyective , that is, the linearized problem has an affirmative answer. If $d\Phi|_{\gamma = \text{const}}$, which is a linear operator, had a closed range, one could conclude that Φ itself is inyective in a sufficiently small neighborhood of γ = const. But the range of $d\Phi$ is not closed and the desired conclusion cannot be obtained in this fashion. Nevertheless, as we shall see below, if γ is sufficiently close to a constant , it is nearly determined by Q_γ and in some cases it can be calculated with an error much smaller than $\|\gamma - \text{const}\|_{L^\infty}$

To show this let us first obtain an expression for the solution of the equation

$$L_\gamma(W) = \nabla.(\gamma\nabla W) = 0 \ , \ W\Big|_{\partial D} = \phi. \quad \gamma = 1 + \delta$$

Let $W = u + v$, where $\Delta u = L_1 u = 0$, $u\Big|_{\partial D} = \phi$. Then

$$L_\gamma(W) = L_{1+\delta}(u+v) = L_1 v + L_\delta v + L_\delta u = 0$$

Since $u\Big|_{\partial D} = W\Big|_{\partial D}$ we have $v\Big|_{\partial D} = 0$ and $v \in H^1_0(D)$, the closure in $H^1(\mathbb{R}^n)$ of functions of C^∞ with support in D. But L_1 , as an operator from $H^1_0(D)$ into $H^{-1}(D)$, has a bounded inverse G, and from the last expression we obtain

$$v + GL_\delta v = -GL_\delta u.$$

and

(1) $$v = -\Big[\sum_0^\infty (-1)^j (GL_\delta)^j\Big](GL_\delta u)$$

Since for $w \in H^1_0(D)$, $\|L_\delta w\|_{H^{-1}(D)} \le \|\delta\|_{L^\infty}\|w\|_{H^1_0(D)}$ if A denotes the norm of G, the series above will converge for $\|\delta\|_{L^\infty}A < 1$ and

(2) $$\|v\|_{H^1(D)} \le \frac{A\|\delta\|_{L^\infty}\|\phi\|}{1 - A\|\delta\|_{L^\infty}}$$

From (1) it follows that Φ is analytic at $\gamma = 1$. The same argument would show that Φ is analytic at any other point γ.

Next let us calculate $d\Phi|_{\gamma=1}$. We have

(3) $$Q_{1+\delta}(\phi) = \int_D (1+\delta)|\nabla w|^2 dx = \int_D \big[(1+\delta)|\nabla u|^2 + 2(\nabla u.\nabla v) + 2\delta(\nabla u.\nabla v) + (1+\delta)|\nabla v|^2\big]dx$$

The contribution of the second term in the integrand of the last integral vanishes on account of the fact that $\Delta u = 0$. Furthermore, from (1) one sees readily that the parts linear in δ of the last two terms in the integrand vanish. Thus setting $d\gamma = \delta$ we obtain

$$dQ_\gamma(\phi)|_{\gamma=1}\delta = \int_D \delta|\nabla u|^2 dx, \quad \Delta u = 0, \quad u|_{\partial D} = \phi$$

To show that $dQ_\gamma|_{\gamma=1}$ is inyective, we merely have to show that if the last integral vanishes for all u with $\Delta u = 0$ then $\delta = 0$. But if the integral vanishes for all such u, then we also have,

(4) $$\int_D \delta(\nabla u_1.\nabla u_2)dx = 0$$

whenever $\Delta u_1 = \Delta u_2 = 0$ in D. Now let Z be any vector in $\mathbb{R}^n$ and a another vector such that $|a| = |Z|$, $(a.Z) = 0$. Then the functions

(5) $$u_1(x) = e^{\pi i(Z.x)+\pi(a.x)}, \quad u_2 = e^{\pi i(Z.x)-\pi(a.x)}.$$

are harmonic, and substituting in (3) we obtain

$$2\pi|Z|^2 \int_D \delta(x) e^{2\pi i(Z.x)}dx = 0, \quad \forall Z$$

whence it follows that $\delta = 0$.

Now let us return to $Q_\gamma(W)$. We set again $\gamma = 1+\delta$ and introduce the bilinear form

$$B(\phi_1,\phi_2) = \frac{1}{2}\big[Q_\gamma(W_1+W_2) - Q_\gamma(W_1) - Q_\gamma(W_2)\big]$$

and setting $W_j = u_j + v_j$, $j = 1,2$, $\Delta u_j = 0$, $u_j|_{\partial D} = \phi_j$ we obtain

$$B(\phi_1,\phi_2) = \int_D (1+\delta)(\nabla u_1.\nabla u_2) + \delta\big[(\nabla u_1.\nabla v_2) + (\nabla u_2.\nabla v_1)\big] + (1+\delta)(\nabla v_1.\nabla v_2)\big]dx.$$

Now, substitution of the exponentials in (5) for u_1 and u_2 in the preceding expression (taking a to be a function of Z such that $|a| = |Z|$, $(a.Z) = 0$) yields

(6) $$\hat\gamma(Z) = \hat F(Z) + R(Z)$$

where $\hat\gamma(Z)$ is the Fourier transform of γ extended to be zero outside D, the function

$$\hat F(Z) = \frac{1}{2\pi^2|Z|^2} B\big(e^{i\pi(Z.x)+\pi(a.x)}, e^{i\pi(Z.x)-\pi(a.x)}\big)$$

is known and, as follows readily from (2),

(7) $$|R(Z)| \leq C\|\delta\|^2_{L^\infty} e^{2\pi r|Z|}$$

provided that $A\|\delta\|_{L^\infty} \leq 1 - \varepsilon$, where C depends only on D and ε, and r is the radius of the smallest sphere containing D. Now R(Z) is too large to permit estimating $\gamma(Z)$. However, under favorable circumstances it is still possible to obtain satisfactory information about γ. Choose α, $1 < \alpha < 2$, then for

(8) $$|Z| \leq \frac{2-\alpha}{\pi\gamma} \log \frac{1}{\|\delta\|_{L^\infty}} = \sigma$$

we have $|R(Z)| \leq C\|\delta\|^\alpha_{L^\infty}$. Let η be a function such that $\hat{\eta} \in C^\infty$, supp $\hat{\eta} \subset \{|Z| \leq 1\}$ $\hat{\eta}(0) = 1$, and let $\eta_\sigma(Z) = \sigma^n \eta(\sigma Z)$. Then we have

$$\hat{\gamma}(Z)\ \hat{\eta}\left(\frac{Z}{\sigma}\right) = \hat{F}(Z)\hat{\eta}\left(\frac{Z}{\sigma}\right) + R(Z)\hat{\eta}\left(\frac{Z}{\sigma}\right)$$

and

(9) $$(\gamma * \eta_\sigma)(x) = (F * \eta_\sigma)(x) + \rho(x)$$

where $*$ denotes convolution and

$$|\rho(Z)| \leq C\|\delta\|^\alpha_{L^\infty} \int \left|\hat{\eta}\left(\frac{Z}{\sigma}\right)\right| dZ =$$

$$= C_1\|\delta\|^\alpha_{L^\infty}\left[\log \frac{1}{\|\delta\|_{L^\infty}}\right]^n$$

where C_1 depends only on D, α and ε.

Thus if $\|\delta\|_{L^\infty}$ is sufficiently small, (9) gives an approximation for $\gamma * \eta_\sigma$ with an erroe which is much smaller than $\|\delta\|_{L^\infty}$. Clearly, if $\|\delta\|_{L^\infty}$ is small then σ is large and $\gamma * \eta_\sigma$ is itself, in some sense, a good approximation to γ.

Approximations to the funcion γ itself be obtained if one assumes that γ, extended to be equal to 1 outside D, is in C^m, $m > n$. In this case one obtains

$$\hat{\delta}(Z) = \hat{F}_1(Z) + R(Z)$$

where F_1 is known and R(Z) is the same as in (6). One then calculates $\delta(x)$ by integrating over $|Z| \leq \sigma$ with σ as in (8) and estimates the error by using the decay of $\hat{\delta}$ at ∞. Thus one obtains

$$\gamma(x) = F_2(x) + \rho(x)$$

where $F_2(x)$ is known and,

$$|\rho(x)| \leq C\|\delta\|^\alpha_{L^\infty}\left[\log \frac{1}{\|\delta\|_{L^\infty}}\right]^n + CM\left[\log \frac{1}{\|\delta\|_{L^\infty}}\right]^{-m+n}$$

where M is a bound for the derivatives of order m of γ.

Bibliography

We have been unable to find treatments of the problem discussed above in the literature, at least not in the general setting in which we are interested. Similar problems have been studied in the papers listed below.

[1] J.R. Cannon, Determination of certain parameters in heat conduction problems, J.Math.Anal. Appl, 8, (1964), p.p. 188-201.

[2] ___________, Determination of an unknown coefficient in a parabolic differential equation, Duke Math. J. (30), (1963), p.p. 313-323.

[3] ___________, Determination of the unknown coefficient k(u) in the equation $\nabla . k(u) \nabla = 0$ from overspecified boundary data, J.Math.Anal, Appl., 18 (1967), p.p. 112-114.

[4] J.R. Cannon and Paul Du Chateau, Determining unknown coefficients in a non-linear heat conduction problem. SIAM J. Appl.Math., 24, n°3 (1973) pp. 298-314.

[5] Jim Douglas, Jr. and G.F. Jones, Determination of a coefficient in a parabolic differential equation, Part II: numerical approximation, J.Math.Mech, 11 (1962) p.p. 919-926.

[6] B.F. Jones , Various methods for finding unknown coefficients in parabolic differential equations, Comm. Pure Appl. Math., 16 (1963) pp. 33-44,

Summary

$\Omega \subset \mathbb{R}^n$ (smooth) bdd domain with boundary Γ.

Relationship between the BVP for the conductivity problem and the BVP for a Schrödinger operator :

$$\mathrm{div}(\gamma \nabla u) = 0 \ \text{ in } \ \Omega, \quad u = \varphi \ \text{ on } \ \Gamma.$$

$$\Downarrow \quad v = \sqrt{\gamma} u \ \ \text{(Liouville transform)}$$

$$(-\Delta + q_\gamma) v = 0 \ \text{ in } \ \Omega, \quad u = \sqrt{\gamma} \varphi \ \text{ on } \ \Gamma.$$

Here

$$q_\gamma = \frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}.$$

Let $q \in L^\infty(\Omega)$ and $\xi \in \mathbb{C}^n$ so that $\xi \cdot \xi = 0$. In that case

$$\Delta e^{-i\xi \cdot x} = 0.$$

Complex Geometric Optic (CGO) solutions : we seek a solution of

$$(-\Delta + q)u = 0 \ \text{ in } \ \Omega$$

of the form

$$u = e^{-i\xi \cdot x}(1 + w_\xi)$$

with

$$\|w\|_{L^2(\Omega)} = O\left(\frac{1}{|\Im \xi|}\right).$$

For $q \in L^\infty$, let $A_q = -\Delta + q$ with domain

$$D(A_q) = H_0^1(\Omega) \cap H^2(\Omega).$$

If $0 \notin \sigma(A_q)$, associate to q the Dirichlet-to-Neumann (DtN) map

$$\Lambda_q : \varphi \in H^{3/2}(\Gamma) \to \partial_\nu u \in H^{1/2}(\Gamma),$$

where $u = u_{q,\varphi} \in H^2(\Omega)$ is the unique variational solution of the BVP

$$(-\Delta + q)u = 0 \ \text{ in } \ \Omega, \quad u = \varphi \ \text{ on } \ \Gamma.$$

Useful identity

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = \int_\Gamma (\Lambda_{q_1} - \Lambda_{q_2})(u_{1|\Gamma}) u_2 d\sigma,$$

for any solutions $u_j \in H^2(\Omega)$ of $(-\Delta + q_j)u_j = 0$ in Ω, $j = 1, 2$.

Uniqueness : $\Lambda_{q_1} = \Lambda_{q_2} \Rightarrow$

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0, \quad u_j \in H^2(\Omega) \text{ and } (-\Delta + q_j)u_j = 0 \text{ in } \Omega. \quad (1)$$

Assume that $n \geq 3$ and fix $k \in \mathbb{R}$. For any $R > 0$, there exist $\xi_1, \xi_2 \in \mathbb{C}^n$ so that

$$\xi_1 + \xi_2 = k, \quad \xi_j \cdot \xi_j = 0, \quad |\Im \xi_j| \geq R, \; j = 1, 2.$$

For R sufficiently large, we can choose u_j in (1) of the form

$$u_j = e^{-i\xi_j \cdot x}(1 + w_j) \quad \text{with} \quad \|w_j\|_{L^2(\Omega)} = O\left(\frac{1}{R}\right). \tag{2}$$

We get

$$\int_\Omega (q_1 - q_2) e^{-ik\cdot x} dx = O\left(\frac{1}{R}\right).$$

Making $R \to \infty$, we obtain

$$\mathscr{F}((q_1 - q_2)\chi_\Omega)(k) = \int_\Omega (q_1 - q_2) e^{-ik\cdot x} dx = 0 \quad \Rightarrow q_1 = q_2.$$

Set $q = (q_1 - q_2)\chi_\Omega$.

Stability : Again, u_j, $j = 1, 2$, given by (2) in (1) yield

$$|\widehat{q}(k)| \leq C\left(\frac{1}{r} + \|\Lambda_{q_1} - \Lambda_{q_2}\| \, e^{Cr}\right), \quad |k| \leq r. \tag{3}$$

We control the high frequencies by assuming an additional estimate of the form

$$\|q\|_{H^s(\mathbb{R}^n)} \leq M, \quad \text{for some } \; s > 0 :$$

$$\int_{|k|\geq R} |\widehat{q}(k)|^2 dk \leq \frac{1}{r^{2s}} \int_{|k|\geq R} \langle k\rangle^{2s} |\widehat{q}(k)|^2 dk \leq \frac{M^2}{r^{2s}}. \tag{4}$$

For instance, if $s = 1$, one gets from (3) and (4)

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C\left(|\ln \|\Lambda_{q_1} - \Lambda_{q_2}\||^{-\frac{2}{n+2}} + \|\Lambda_{q_1} - \Lambda_{q_2}\|\right).$$

Back to the inverse conductivity problem : Let $\gamma \in W^{2,\infty}(\Omega)$ satisfying $\gamma \geq \gamma_0$, for some constant $\gamma_0 > 0$. We associate to γ the DtN map

$$\Lambda_\gamma : \varphi \in H^{3/2}(\Gamma) \to \gamma \partial_\nu u \in H^{1/2}(\Gamma),$$

where $u = u_{\gamma,\varphi} \in H^2(\Omega)$ is the unique solution of the BVP

$$\mathrm{div}(\gamma \nabla u) = 0 \ \text{ in } \ \Omega \ \text{ and } \ u = \varphi \ \text{ on } \ \Gamma.$$

The map Λ_γ is connected to Λ_{q_γ}, where $q_\gamma = \gamma^{-1/2}\Delta\gamma^{1/2}$, by the formula

$$\Lambda_\gamma = \frac{1}{2}\gamma^{-1}\partial_\nu \gamma I + \gamma^{-1/2}\Lambda_{q_\gamma}\gamma^{-1/2}.$$

We firstly need to determine γ and $\nabla\gamma$ on Γ. To do that we employ singular solutions having singularities localized near the boundary. The scheme of the proof is

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2, \ \ \nabla\gamma_1 = \nabla\gamma_2 \ \text{ on } \ \Gamma$$
$$\Rightarrow \Lambda_{q_{\gamma_1}} = \Lambda_{q_{\gamma_2}} \Rightarrow q_{\gamma_1} = q_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2.$$

Some literature

Uniqueness for the global DtN map :

Sylvester-Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. Math. 125 (1987), 153-169.

Log stability for the global DtN map :

Alessandrini, Stable determination of conductivity by boundary measurements, Appl. Anal. 27 (1988), 153-172.

The log stability is the best possible :

Mandache, Exponential instability in an inverse problem for the Schrödinger equation, Inverse Problems 17 (2001), 1435-1444.

Partial DtN map :

Bukhgeim-Uhlmann, Recovering a potential from partial Cauchy data, Commun. Part. Diff. Equat. 27 (2002), 653-658.

Kenig-Sjöstrand-Uhlmann, The Calderòn problem with partial data , Ann. Math. 165 (2007), 567-791.

Reconstruction :

Nachman and B. Street, Reconstruction in the Calderòn problem with partial data, Commun. Part. Diff. Equat. 35 (2010), 375-390.

Stability of a partial DtN map :

Heck and J.-N. Wang, Stability estimate for the inverse boundary value problem by partial Cauchy data, Inv. Probl. 22 (2006), 1787-1797.

Caro-Dos Santos Ferreira-Ruiz, Stability estimates for the Calderón problem with partial data, JDE 2016.

Summary

Recall that, for $q \in L^\infty(\Omega)$,

$$A_q = -\Delta + q \ \text{ with } \ D(A_q) = H_0^1(\Omega) \cap H^2(\Omega).$$

Let

$$\mathcal{Q} = \{q \in L^\infty(\Omega; \mathbb{R});\ 0 \notin \sigma(A_q)\}.$$

For any $q \in \mathcal{Q}$ and $\varphi \in H^{-1/2}(\Gamma)$, the BVP

$$(-\Delta + q)u = 0 \ \text{ in } \ \Omega, \quad u = \varphi \ \text{ on } \ \Gamma$$

admits a unique (transposition) solution $u_{q,\varphi} \in H_\Delta(\Omega)$, where

$$H_\Delta(\Omega) = \{u \in L^2(\Omega);\ \Delta u \in L^2(\Omega)\}.$$

Lemma 1 (trace theorem)

For $j = 0, 1$, the trace map

$$t_j u = \partial_\nu^j u_{|\Gamma}, \;\; u \in \mathscr{D}(\overline{\Omega}),$$

extends to a continuous operator, still denoted by t_j, from $H_\Delta(\Omega)$ into $H^{-j-1/2}(\Gamma)$. Namely, there exists $c_j > 0$, such that the estimate

$$\|t_j u\|_{H^{-j-1/2}(\Gamma)} \leq c_j \|u\|_{H_\Delta(\Omega)},$$

holds for every $u \in H_\Delta(\Omega)$.

Additionally we get with the help of Lemma 1 that the mapping

$$\Lambda_q : \varphi \in H^{-1/2}(\Gamma) \to \partial_\nu u_{q,\varphi} \in H^{-3/2}(\Gamma)$$

defines a bounded operator.

Remark : By employing the method of Lee-Uhlamnn, one can check that Λ_q is ΨDO of order 1 ; while $\Lambda_{q_1,q_2} = \Lambda_{q_1} - \Lambda_{q_2}$ is a ΨDO of order -1 :

$$\Lambda_{q_1,q_2} \in \mathscr{B}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma)).$$

Set, for $\xi \in \mathbb{S}^{n-1}$,

$$\Gamma_{\pm,\xi} = \{x \in \Gamma;\ \pm\xi \cdot \nu(x) > 0\}.$$

Let F (resp. G) an open neighborhood of $\Gamma_{+,\xi}$ (resp. $\Gamma_{-,\xi}$) in Γ. Define

$$\widetilde{\Lambda}_{q_1,q_2} : \varphi \in H^{-1/2}(\Gamma) \cap \mathscr{E}'(F) \to \Lambda_{q_1,q_2}(\varphi)_{|G}.$$

This operator is bounded from $H^{-1/2}(\Gamma) \cap \mathscr{E}'(F)$, endowed with the norm of $H^{-1/2}(\Gamma)$, into $H^{1/2}(G)$.

The norm of $\widetilde{\Lambda}_{q_1,q_2}$ in $\mathscr{B}(H^{-1/2}(\Gamma) \cap \mathscr{E}'(F), H^{1/2}(G))$ is denoted by $\|\widetilde{\Lambda}_{q_1,q_2}\|$.

Theorem 1 (Choulli-Kian-Soccorsi '15)

For any $\delta > 0$ and $t > 0$, there exists a constant $C > 0$, depending only on δ and t, so that

$$\|q_1 - q_2\|_{L^2(\Omega)} \leq C \left(\|\widetilde{\Lambda}_{q_1,q_2}\| + \left| \ln \left| \ln \|\widetilde{\Lambda}_{q_1,q_2}\| \right| \right|^{-t} \right),$$

for any $q_1, q_2 \in \mathcal{Q} \cap \delta B_{L^\infty(\Omega)}$ satisfying $(q_2 - q_1)\chi_\Omega \in \delta B_{H^t(\mathbb{R}^n)}$, and

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq C \left(\|\widetilde{\Lambda}_{q_1,q_2}\| + \left| \ln \left| \ln \|\widetilde{\Lambda}_{q_1,q_2}\| \right| \right|^{-1} \right),$$

for any $q_1, q_2 \in \mathcal{Q} \cap \delta B_{L^2(\Omega)}$.

Here and henceforth B_X is the unit ball of the Banach space X.

Summary

Proposition 1

For $\delta > 0$ fixed, let $q \in \delta B_{L^\infty(\Omega)}$. Let $\zeta, \eta \in \mathbb{S}^{n-1}$ satisfy $\zeta \cdot \eta = 0$ and fix $\epsilon > 0$ so small that $\Gamma_-^\epsilon = \Gamma_-^\epsilon(\zeta) = \{x \in \Gamma;\ \zeta \cdot \nu(x) < -\epsilon\} \neq \emptyset$. There exists $\tau_0 = \tau_0(\delta) > 0$, so that the BVP

$$\begin{cases} (-\Delta + q)u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_-^\epsilon. \end{cases}$$

has a solution of the form $u = e^{\tau(\zeta + i\eta)\cdot x}(1+\psi) \in H_\Delta(\Omega)$ with ψ obeying

$$\|\psi\|_{L^2(\Omega)} \leq C\tau^{-1/2},$$

for some constant $C > 0$ depending only on δ, Ω and ϵ.

The key point in the proof of this lemma is a Carleman inequality by Bukhgeim-Uhlamnn : there exist $\tau_0 = \tau_0(\delta) > 0$ and $C = C(\delta) > 0$ so that

$$C\tau^2 \int_\Omega e^{-2\tau x\cdot\zeta}|v|^2 dx + \tau \int_{\Gamma_+} |\zeta\cdot\nu(x)|e^{-2\tau x\cdot\zeta}|\partial_\nu v|^2 d\sigma$$
$$\leq \int_\Omega e^{-2\tau x\cdot\zeta}|(\Delta - q)v|^2 dx + \tau \int_{\Gamma_-} |\zeta\cdot\nu(x)|e^{-2\tau x\cdot\zeta}|\partial_\nu v|^2 d\sigma$$

holds for all $\tau \geq \tau_0$ and $v \in H_0^1(\Omega) \cap H^2(\Omega)$.

Summary

Denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$.

Let u_j be a CGO solution given by Proposition 1 and corresponding to $q_j \in \delta B_{L^\infty(\Omega)}$, $j = 1, 2$.

Generalized Green's formula yields

$$\int_\Omega (q_2 - q_1) u_1 u_2 dx = \langle t_1 \overline{\Lambda_{q_1, q_2}(t_0 u_2)}, t_0 u_1 \rangle.$$

The estimate in Proposition 1 implies : there exist a subset E of $\mathbb{S}^{n-1}$ with $|E| > 0$ so that

$$\left| \int_\Omega (q_2 - q_1) e^{-i\kappa \cdot x} dx \right| \leq C \left(e^{2d\tau} \|\widetilde{\Lambda}_{q_1, q_2}\| + \tau^{-1/2} \right), \qquad (5)$$

holds uniformly in $\kappa \in rE$ and $r \in (0, 2\tau)$.

Theorem 2 (Apraiz-Escauriaza-Wang-Zhang)

Assume that $F : 2\mathbb{B}^n \to \mathbb{C}$ *is real-analytic and satisfies*

$$|\partial^\alpha F(\kappa)| \leq K \frac{|\alpha|!}{\rho^{|\alpha|}},\ \kappa \in 2\mathbb{B},\ \alpha \in \mathbb{N}^n,$$

for some $(K, \rho) \in \mathbb{R}_+^* \times (0, 1]$. *Then for any measurable set* $E \subset \mathbb{B}$ *with* $|E| > 0$, *there exist two constants* $M = M(\rho, |E|) > 0$ *and* $\theta = \theta(\rho, |E|) \in (0, 1)$ *such that*

$$\|F\|_{L^\infty(\mathbb{B})} \leq MK^{1-\theta} \left(\frac{1}{|E|} \int_E |F(\kappa)| d\kappa \right)^\theta .$$

(5) + Theorem 2 $\Rightarrow$

$$|\widehat{q}(r\kappa)| \leq Ce^{(1-\theta)r}\left(e^{d\tau}\|\widetilde{\Lambda}_{q_1,q_2}\| + \tau^{-1/2}\right)^{\theta},\ \kappa \in \mathbb{B},$$

Combine this with an estimate for high frequencies in order to derive

$$\|q\|^2_{L^2(\Omega)} \leq Cr^n e^{2(1-\theta)r}\left(e^{d\tau}\|\widetilde{\Lambda}_{q_1,q_2}\| + \tau^{-1/2}\right)^{2\theta} + \frac{M^2}{r^{2t}},$$

$r \in (0, 2\tau),\ \tau \in [\tau_0, +\infty)$. Whence

$$\|q\|^2_{L^2(\Omega)} \leq C' e^{(n+2)r}\left|\ln\|\widetilde{\Lambda}_{q_1,q_2}\|\right|^{-\theta} + \frac{M^2}{r^{2t}},\ r \in (0, 2\tau_*).$$

The proof is completed by a minimizing with respect to r.

Summary

If $\sigma \in W_+^{1,\infty}(\Omega) = \{c \in W^{1,\infty}(\Omega;\mathbb{R});\ c(x) \geq c_0 \text{ for some } c_0 > 0\}$, introduce the Hilbert space

$$H_{\mathrm{div}(\sigma\nabla)}(\Omega) = \{u \in L^2(\Omega),\ \mathrm{div}(\sigma\nabla u) \in L^2(\Omega)\}$$

endowed with the norm

$$\|u\|_{H_{\mathrm{div}(\sigma\nabla)}(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\mathrm{div}(\sigma\nabla u)\|^2_{L^2(\Omega)}\right)^{1/2}.$$

By a slight modification of the proof of Lemma 1 the trace map

$$t_j^\sigma u = \sigma^j \partial_\nu^j u_{|\Gamma},\ u \in \mathscr{D}(\overline{\Omega}),\ j = 0, 1,$$

is extended to a linear continuous operator, still denoted by t_j^σ, from $H_{\mathrm{div}(\sigma\nabla)}(\Omega)$ into $H^{-j-1/2}(\Gamma)$.

For $g \in H^{-1/2}(\Gamma)$, the mapping

$$\Lambda_\sigma : g \in H^{-1/2}(\Gamma) \mapsto t_1^\sigma u_{\sigma,g} \in H^{-3/2}(\Gamma)$$

defines a bounded operator, where $u_{\sigma,g} \in H_{\mathrm{div}(\sigma\nabla)}(\Omega)$ is the unique (transposition) solution of the BVP

$$\mathrm{div}(\sigma \nabla u) = 0 \text{ in } \Omega, \quad u = g \text{ on } \Gamma,$$

Recall that

$$\Lambda_{q_\sigma} = \frac{1}{2}\sigma^{-1}(\partial_\nu \sigma) I + \sigma^{-1/2}\Lambda_\sigma \sigma^{-1/2}, \tag{6}$$

with $q_\sigma = \sigma^{-1/2}\Delta\sigma^{1/2}$.

Let

$$\widetilde{\Lambda}_{\sigma_1,\sigma_2} : g \in H^{-1/2}(\Gamma) \cap \mathscr{E}'(F) \mapsto (\Lambda_{\sigma_1} - \Lambda_{\sigma_2})(g)_{|G} \in H^{1/2}(G).$$

Then, with reference to (6),

$$\widetilde{\Lambda}_{q_1,q_2} g = \sigma_1^{-1/2} \widetilde{\Lambda}_{\sigma_1,\sigma_2}(\sigma_1^{-1/2} g),$$

for every $g \in H^{-1/2}(\Gamma) \cap \mathscr{E}'(F)$, provided

$$\sigma_1 = \sigma_2 \ \text{ on } \ \Gamma \ \text{ and } \ \partial_\nu \sigma_1 = \partial_\nu \sigma_2 \ \text{ on } \ F \cap G.$$

Therefore

$$\|\widetilde{\Lambda}_{q_1,q_2}\| \leq C \|\widetilde{\Lambda}_{\sigma_1,\sigma_2}\|, \tag{7}$$

where $\|\cdot\|$ still denotes the norm of $\mathscr{B}(H^{-1/2}(\Gamma) \cap \mathscr{E}'(F), H^{1/2}(G))$.

Taking into account that $\phi = \sigma_1^{1/2} - \sigma_2^{1/2}$ is solution The BVP

$$\begin{cases} (-\Delta + q_1)\phi = \sigma_2^{1/2}(q_2 - q_1) & \text{in } \Omega \\ \phi = 0 & \text{on } \Gamma, \end{cases}$$

we prove

$$\|\sigma_1^{1/2} - \sigma_2^{1/2}\|_{L^2(\Omega)} \leq C\|q_2 - q_1\|_{H^{-1}(\Omega)}$$

and then

$$\|\sigma_1 - \sigma_2\|_{L^2(\Omega)} \leq C\|q_2 - q_1\|_{H^{-1}(\Omega)}. \tag{8}$$

(7) +(8) +Theorem 1 $\Rightarrow$

Corollary 3 (Choulli-Kian-Soccorsi '15)

Let $\delta > 0$ and $\sigma_0 > 0$. Then for any $\sigma_j \in \delta B_{W^{2,\infty}(\Omega)}$, $j = 1, 2$, obeying $\sigma_j \geq \sigma_0$ and

$$\sigma_1 = \sigma_2 \text{ on } \Gamma \text{ and } \partial_\nu \sigma_1 = \partial_\nu \sigma_2 \text{ on } F \cap G,$$

we may find a constant $C > 0$, independent of σ_1 and σ_2, so that

$$\|\sigma_1 - \sigma_2\|_{L^2(\Omega)} \leq C \left(\|\widetilde{\Lambda}_{\sigma_1,\sigma_2}\| + \left| \ln \left| \ln \|\widetilde{\Lambda}_{\sigma_1,\sigma_2}\| \right| \right|^{-1} \right).$$

Summary

Let Ω be a C^2-bdd domain of $\mathbb{R}^n$, $n \geq 2$, with boundary Γ and, for $T > 0$, set

$$Q = \Omega \times (0, T), \quad \Omega_+ = \Omega \times \{0\}, \quad \Sigma = \Gamma \times (0, T).$$

Consider the IBVP

$$\begin{cases} (\partial_t - \Delta + q(x,t))u = 0 & \text{in } Q, \\ u_{|\Omega_+} = 0, \\ u_{|\Sigma} = g. \end{cases} \tag{9}$$

Following Lions and Magenes, $H^{-r,-s}(\Sigma)$, $r, s > 0$, denotes the dual space of

$$H^{r,s}_{,0}(\Sigma) = L^2(0, T; H^r(\Gamma)) \cap H^s_0(0, T; L^2(\Gamma)).$$

For $q \in L^\infty(Q)$ and $g \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$, the IBVP (9) admits a unique (transposition) solution $u_{q,g} \in L^2(Q)$. Additionally the following parabolic DtN map

$$\Lambda_q : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{-\frac{3}{2},-\frac{3}{4}}(\Sigma)$$
$$g \mapsto \partial_\nu u_{q,g}$$

is bounded.

Recall that, for $\omega \in \mathbb{S}^{n-1}$,

$$\Gamma_{\pm,\omega} = \{x \in \Gamma;\ \pm\nu(x) \cdot \omega > 0\}$$

and set

$$\Sigma_{\pm,\omega} = \Gamma_{\pm,\omega} \times (0, T).$$

Fix $\omega_0 \in \mathbb{S}^{n-1}$, $\mathcal{U}_\pm$ a neighborhood of $\Gamma_{\pm,\omega_0}$ in Γ and set

$$\mathcal{V}_+ = \mathcal{U}_+ \times [0,T], \quad \mathcal{V}_- = \mathcal{U}_- \times (0,T).$$

Define then the partial parabolic DtN operator

$$\begin{aligned} \widehat{\Lambda}_q : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \cap \mathscr{E}'(\mathcal{V}_+) &\to H^{-\frac{3}{2},-\frac{3}{4}}(\mathcal{V}_-) \\ g &\mapsto \partial_\nu u_{q,g}{}_{|\mathcal{V}_-}. \end{aligned}$$

Observe that as in the elliptic case $\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}}$ is a smoothing operator : $\widehat{\Lambda}_q - \widehat{\Lambda}_{\widetilde{q}} \in \mathscr{B}(H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma), H^{\frac{1}{2},\frac{1}{4}}(\Sigma))$.

For $\frac{1}{2(n+3)} < s < \frac{1}{2(n+1)}$, set

$$\Psi_s(\rho) = \rho + |\ln \rho|^{-\frac{1-2s(n+1)}{8}}, \quad \rho > 0, \tag{10}$$

extended by continuity at $\rho = 0$ by setting $\Psi_s(0) = 0$.

Theorem 4 (Choulli-Kian '16)

Fix $\delta > 0$ and $\frac{1}{2(n+3)} < s < \frac{1}{2(n+1)}$. There exists a constant $C > 0$, that can depend only on δ, Q and s, so that, for any $q_1, q_2 \in \delta B_{L^\infty(Q)}$,

$$\|q_1 - q_2\|_{H^{-1}(Q)} \leq C\Psi_s\left(\|\Lambda_{q_1} - \Lambda_{q_2}\|\right). \tag{11}$$

Here $\|\Lambda_{q_1} - \Lambda_{q_2}\|$ stands for the norm of $\Lambda_{q_1} - \Lambda_{q_2}$ in $\mathscr{B}(H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma); H^{\frac{1}{2},\frac{1}{4}}(\Sigma))$.

Set

$$\Phi_s(\rho) = \rho + |\ln|\ln\rho||^{-s}, \quad \rho > 0,\ s > 0, \tag{12}$$

extended by continuity at $\rho = 0$ by setting $\Phi_s(0) = 0$.

Theorem 5 (Choulli-Kian '16)

Let $\delta > 0$, there exist two constants $C > 0$ and $s \in (0, 1/2)$, that can depend only on δ, Q and $\mathcal{V}_\pm$, so that, for any $q_1, q_2 \in \delta B_{L^\infty(Q)}$,

$$\|q_1 - q_2\|_{H^{-1}(Q)} \leq C\Phi_s\left(\|\widehat{\Lambda}_{q-1} - \widehat{\Lambda}_{q_2}\|\right). \tag{13}$$

Here $\|\widehat{\Lambda}_q - \widehat{\Lambda}_{\tilde{q}}\|$ denotes the norm of $\widehat{\Lambda}_q - \widehat{\Lambda}_{\tilde{q}}$ in $\mathscr{B}(H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma); H^{\frac{1}{2},\frac{1}{4}}(\mathcal{V}_-))$.

Thank you for your attention