# Numerical approximation of boundary controls for beam equations

**Nicolae Cîndea**
joint work with Sorin Micu and Ionel Roventa

Valenciennes, July 4 – 7, 2016
Stability of non-conservative systems

We consider the following clamped beam equation :

$$\begin{cases} \ddot{u}(x,t) + \partial_x^4 u(x,t) = 0, & (0,1)\times(0,T) \\ u(0,t) = u(1,t) = 0, & t\in(0,T) \\ \partial_x u(0,t) = 0, \quad \partial_x u(1,t) = v(t), & t\in(0,T) \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = u_1(x), & x\in(0,1). \end{cases} \qquad \text{(CS)}$$

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We say that the beam equation (CS) is null controllable in time $T > 0$, if for every initial data $(u_0, u_1) \in L^2(0,1) \times H^{-2}(0,1)$ there exists a control $v \in L^2(0,T)$ such that

$$u(\cdot, T) = \dot{u}(\cdot, T) = 0.$$

# Controllability of the Euler-Bernoulli beam equation

We consider the following **hinged beam equation**:

$$\begin{cases} \ddot{u}(x,t) + \partial_x^4 u(x,t) = 0, & (0,1)\times(0,T) \\ u(0,t) = u(1,t) = 0, & t\in(0,T) \\ \partial_x^2 u(0,t) = 0, \quad \partial_x^2 u(1,t) = v(t), & t\in(0,T) \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = u_1(x), & x\in(0,1). \end{cases} \tag{CS}$$

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# Observability of the beam equation

In order to define the dual observability concept, we consider the following homogeneous clamped beam equation:

$$\begin{cases} \ddot{y}(x,t) + \partial_x^4 y(x,t) = 0, & (0,1)\times(0,T) \\ y(0,t) = y(1,t) = 0, & t\in(0,T) \\ \partial_x y(0,t) = \partial_x y(1,t) = 0, & t\in(0,T) \\ y(x,0) = y_0(x), \quad \dot{y}(x,0) = y_1(x), & x\in(0,1). \end{cases} \qquad \text{(S)}$$

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We say that the beam equation (S) is exactly observable in time $T > 0$, if there exists a constant $K_T > 0$ such that for every initial data $(y_0, y_1) \in H_0^2(0,1) \times L^2(0,1)$ the solution $y$ satisfies

$$\|y_0\|^2_{H_0^2(0,1)} + \|y_1\|^2_{L^2(0,1)} \leq K_T \int_0^T |\partial_x^2 y(1,t)|^2 dt \tag{OBS}$$

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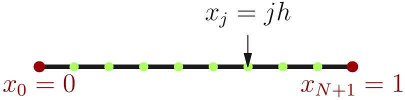
# Finite differences semi-discretization

$N$ discretization points in $(0, 1)$

$$h = \frac{1}{N+1}$$



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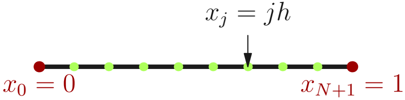


$$\partial_x^4 u(x_j, t) \approx \frac{u(x_{j-2}, t) - 4u(x_{j-1}, t) + 6u(x_j, t) - 4u(x_{j+1}, t) + u(x_{j+2}, t)}{h^4}$$

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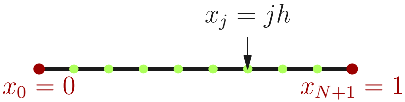


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clamped beam

$$A = \begin{pmatrix} 7 & -4 & 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ -4 & 6 & -4 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & \dots & \dots & \dots & 0 & 1 & -4 & 6 & -4 \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 & -4 & 7 \end{pmatrix}$$

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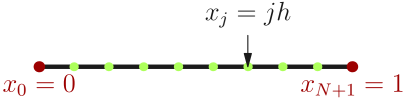


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\end{pmatrix}$$

The following semi-discrete finite-dimensional system is an approximation of the clamped beam equation (CS)

$$\begin{cases} \ddot{U}_h(t) + A_h U_h(t) = F_h(t), & t \in (0,T) \\ U_h(0) = U_h^0, \quad \dot{U}_h(0) = U_h^1, \end{cases} \tag{CS$_h$}$$

where $A_h = \dfrac{1}{h^4} A$ and

$$U_h^i = \begin{pmatrix} u_1^i \\ u_2^i \\ \vdots \\ u_N^i \end{pmatrix}, \; U_h(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{pmatrix} \quad F_h(t) = -\frac{1}{h^3} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v_h(t) \end{pmatrix}.$$

clamped beam

## Discrete controllability problem

For a given time $T > 0$ and for every initial data $(U_h^0, U_h^1) \in \mathbb{C}^N \times \mathbb{C}^N$ find a control $v_h \in L^2(0,T)$ such that

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# A uniform observability inequality?

**Aim:** to study the discrete observability property corresponding to the controlled problem ($\mathsf{CS}_h$) which reads as follows: there exists a constant $K_h$ such that the following inequality holds

clamped beam

$$\|Y_h^0\|_2^2 + \|Y_h^1\|_0^2 \leq K_h \int_0^T \left| \frac{Y_{hN}(t)}{h^2} \right|^2 dt, \qquad (\mathsf{OBS}_h)$$

for any $\begin{pmatrix} Y_h^0 \\ Y_h^1 \end{pmatrix} \in \mathbb{C}^{2N}$, where $\begin{pmatrix} Y_h \\ \dot{Y}_h \end{pmatrix}$ is the solution of the following semi-discretization of (S)

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## Question

The constant $K_h$ is uniformly bounded w.r.t. $h$?

# The case of the hinged beam equation

L. León, E. Zuazua, *Boundary controllability of the finite-difference space semi-discretizations of the beam equation.* ESAIM COCV, 2002, 8, 827-862.

- explicit form of the eigenvalues and eigenvectors of the matrix $A$
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$\Rightarrow$ uniform observability

  - filtering of the high-frequencies at the level $\gamma h^{-4}$ for $\gamma \in (0,1)$
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I.F. Bugariu, S. Micu,; I. Roventa, *Approximation of the controls for the beam equation with vanishing viscosity.* Math. Comp. 85 (2016), no. 301, 2259–2303.

- adding a viscous term of the form $\varepsilon A_h \dot{Y}_h$ with $\varepsilon \in (\frac{h^2}{2T} \ln(h^{-1}), h)$
- moment method

$\Rightarrow$ uniform controllability

## Theorem (NC, S. Micu, I.Rovența)

*Let $T > 0$ and $\gamma \in (0,1)$. There exists $N_0 \in \mathbb{N}$ such that for every $N \geq N_0$ the observability inequality* (OBS$_h$) *holds, with a positive constant $K$ independent of $h$, for every solution of* (S) *with initial data in the space $C_h(\gamma)$. Moreover,*

$$\lim_{h\to\infty} \sup \left\{ \frac{\|Y_h^0\|_2^2 + \|Y_h^1\|_0^2}{\displaystyle\int_0^T \left|\frac{Y_{hN}(t)}{h^2}\right|^2 dt} \;\middle|\; \begin{array}{l} \begin{pmatrix} Y_h^0 \\ Y_h^1 \end{pmatrix} \in \mathbb{C}^{2N} \text{ and} \\ \begin{pmatrix} Y_h \\ \dot{Y}_h \end{pmatrix} \text{ solution of } (\mathsf{S}_h) \end{array} \right\} = \infty.$$

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$$\boxed{A\phi^n = \lambda_n \phi^n} \qquad \boxed{\Phi^n = \begin{pmatrix} \frac{h^2}{\sqrt{\lambda_{|n|}}} \\ -\mathrm{sgn}(n)\, i \end{pmatrix} \phi^{|n|}}$$

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similar results for hinged and clamped beam


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- Abstract systems case


S. Ervedoza, *Spectral conditions for admissibility and observability of wave systems: applications to finite element schemes.* Numer. Math., 2009, 113, 377-415

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filtering at the range $Ch^{-\frac{4}{3}+\varepsilon}$

## Proposition

*The matrix $A$ has only real eigenvalues $(\lambda_n)_{1\leq n\leq N} \subset (0,16)$ and there exists an orthonormal basis in $\mathbb{C}^N$ (with respect to the canonical inner product $\langle\cdot,\ \cdot\rangle_0$) consisting of eigenvectors $(\phi^n)_{1\leq n\leq N}$ of $A$.*

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**hinged beam**

$$\lambda_n = 16\sin^4\left(\frac{n\pi h}{2}\right)$$

$$\phi_j^n = \sin(jn\pi h)$$

## Proposition

*With the above notation, $\lambda$ is a eigenvalue of the matrix $A$ if and only if verifies one of the following relations*

$$\cos\left((N+1)\arg(X_4)\right) = \frac{8X_1^{N+1} - \sqrt{\lambda}X_1^{2(N+1)} - \sqrt{\lambda}}{2\left(2X_1^{2(N+1)} - \sqrt{\lambda}X_1^{N+1} + 2\right)}, \qquad \sin((N+1)\arg(X_4)) > 0,$$

*or*

$$\cos\left((N+1)\arg(X_4)\right) = \frac{8X_1^{N+1} + \sqrt{\lambda}X_1^{2(N+1)} + \sqrt{\lambda}}{2\left(2X_1^{2(N+1)} + \sqrt{\lambda}X_1^{N+1} + 2\right)}, \qquad \sin((N+1)\arg(X_4)) < 0,$$

*where for each $j \in \{1,\ 2,\ 3,\ 4\}$ the numbers $X_j$ are given by*

$$X_{1,2} = \frac{2+\sqrt{\lambda} \pm \sqrt{(2+\sqrt{\lambda})^2 - 4}}{2}, \ X_{3,4} = \frac{2-\sqrt{\lambda} \pm i\sqrt{4-(2-\sqrt{\lambda})^2}}{2}.$$

The proof of the proposition is somehow similar the same as for the discrete Laplacian in the the book of Keller and Isaacson:

- $n$-th line of linear system $A\phi = \lambda\phi$

$$\phi_{n+2} - 4\phi_{n+1} + (6-\lambda)\phi_n - 4\phi_{n-1} + \phi_{n-2} = 0$$

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- $X_i$ $(i \in \{1,2,3,4\})$ are the solutions of

$$x^4 - 4x^3 + (6-\lambda)x^2 - 4x + 1 = 0.$$

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$$\phi_n = C_1 X_1^n + C_2 X_2^n + C_3 X_3^n + C_4 X_4^n$$

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- components of the eigenvector $\phi$ write as

$$\phi_n = C_1 X_1^n + C_2 X_2^n + C_3 X_3^n + C_4 X_4^n$$

- boundary conditions on $\phi$

$$\phi_0 = \phi_{N+1} = 0$$

$$\phi_{-1} = \phi_1, \qquad \phi_N = \phi_{N+2}$$

$$
\left\{
\begin{array}{rcrcrcrcl}
C_1 & + & C_2 & + & C_3 & + & C_4 & = & 0 \\
R_+ C_1 & - & R_+ C_2 & + & iR_- C_3 & - & iR_- C_4 & = & 0 \\
X_1^{N+1} C_1 & + & X_2^{N+1} C_2 & + & X_3^{N+1} C_3 & + & X_4^{N+1} C_4 & = & 0 \\
X_1^{N+1} R_+ C_1 & - & X_2^{N+1} R_+ C_2 & + & iX_3^{N+1} R_- C_3 & - & iX_4^{N+1} R_- C_4 & = & 0.
\end{array}
\right.
$$

From the first two equations we extract

$$
\begin{aligned}
C_3 &= -\frac{1}{2}\left(1 - i\frac{R_+}{R_-}\right) C_1 - \frac{1}{2}\left(1 + i\frac{R_+}{R_-}\right) C_2, \\
C_4 &= -\frac{1}{2}\left(1 + i\frac{R_+}{R_-}\right) C_1 - \frac{1}{2}\left(1 - i\frac{R_+}{R_-}\right) C_2,
\end{aligned}
$$

and from the last two equations

$$
\begin{aligned}
C_3 &= -\frac{1}{2}\left(1 - i\frac{R_+}{R_-}\right) \frac{X_1^{N+1}}{X_3^{N+1}} C_1 - \frac{1}{2}\left(1 + i\frac{R_+}{R_-}\right) \frac{X_2^{N+1}}{X_3^{N+1}} C_2, \\
C_4 &= -\frac{1}{2}\left(1 + i\frac{R_+}{R_-}\right) \frac{X_1^{N+1}}{X_4^{N+1}} C_1 - \frac{1}{2}\left(1 - i\frac{R_+}{R_-}\right) \frac{X_2^{N+1}}{X_4^{N+1}} C_2,
\end{aligned}
$$

- any number $\lambda \in (0, 16)$ can be written as

$$\lambda = 16 \sin^4 \left( \frac{hz}{2} \right)$$

for some $z \in \left(0, \frac{\pi}{h}\right)$ and, hence, $\arg(X_4) = 2\pi - zh$.

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- the new variable $z$ satisfies the equations

$$f^{\pm}(z) := g^{\pm}(z) - \frac{2\left(1-\sin^4\left(\frac{hz}{2}\right)\right) r^{N+1}(z)}{r^{2(N+1)}(z) \mp 2\sin^2\left(\frac{hz}{2}\right) r^{N+1}(z) + 1} = 0,$$

where

$$g^{\pm}(z) = \cos(z) \pm \sin^2(\frac{zh}{2}).$$

$$r(z) = 1 + 2\sin^2\left(\frac{zh}{2}\right) + 2\sqrt{\sin^2\left(\frac{zh}{2}\right)\left(1+\sin^2\left(\frac{zh}{2}\right)\right)}.$$

# Characterization of the high-frequencies

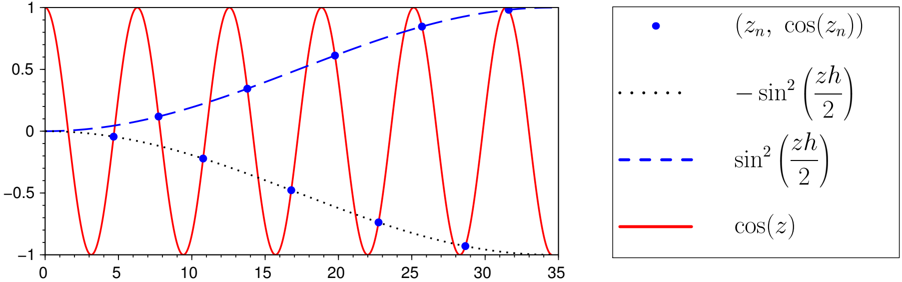
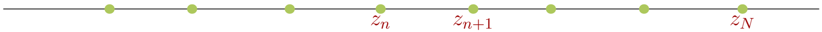
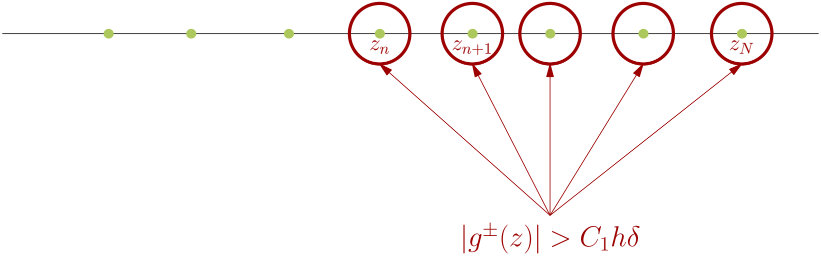
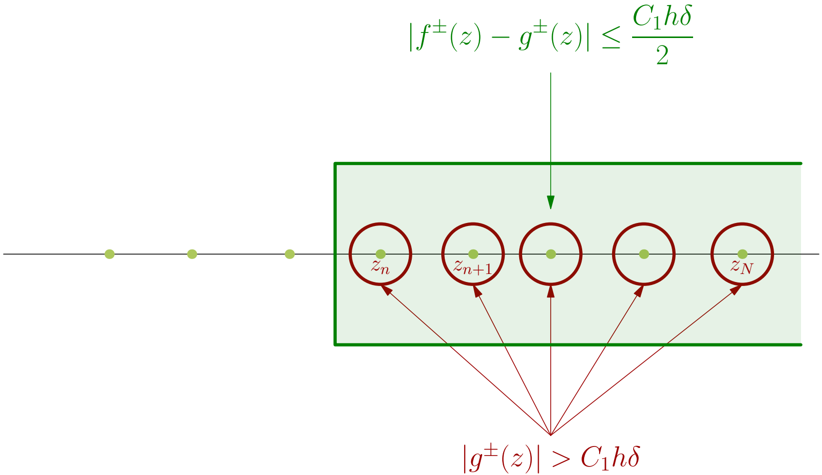


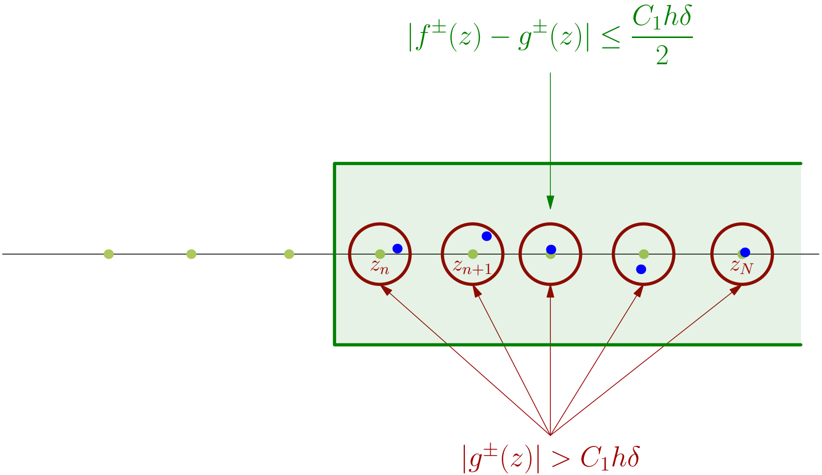

Figure: Solutions $z_n$ of equations $g^{\pm}(z) = 0$ for $N = 10$.

$$g^{\pm}(z) = \cos(z) \pm \sin^2(\frac{zh}{2}).$$


$z_n$
$z_{n+1}$
$z_N$


$z_n$
$z_{n+1}$
$z_N$
$|g^{\pm}(z)| > C_1 h\delta$


$|f^{\pm}(z) - g^{\pm}(z)| \leq \frac{C_1 h\delta}{2}$
$z_n$
$z_{n+1}$
$z_N$
$|g^{\pm}(z)| > C_1 h\delta$




By Rouché's Theorem, if $N$ and $n$ are large enough,
the zeros $y_n^{\pm}$ of $f^{\pm}$ are close to zeros $z_n^{\pm}$ of $g^{\pm}$.

## Proposition (NC, S. Micu, I. Rovența)

*Let* $\varrho > 1$. *There exists* $\delta_0 > 0$ *such that, for each* $\delta \in (0, \delta_0)$, *there exists* $N_0(\delta) \in \mathbb{N}^*$ *with the property that the eigenvalues* $(\lambda_n)_{\varrho \ln N \leq n \leq N}$ *of the matrix* $A \in \mathcal{M}_N(\mathbb{R})$ *with* $N \geq N_0(\delta)$ *are given by*

$$\lambda_n = \begin{cases} 16 \sin^4 \left( \frac{y_k^+ h}{2} \right) & \text{if } n = 2k+2, \\ 16 \sin^4 \left( \frac{y_k^- h}{2} \right) & \text{if } n = 2k+1, \end{cases}$$

*where* $y_k^+$ *and* $y_k^-$ *are zeros of the functions* $f^+$ *and* $f^-$.

# Observability of the high-order eigenvectors

**Theorem (N.C., S. Micu, I. Rovenţa)**

*Let $\sigma \in (0,1)$. There exist $K > 0$ and $N_0 \in \mathbb{N}^*$ such that, for each $N \geq N_0$ and each $\lambda$ eigenvalue of the matrix $A$ with the property that $\lambda \in (\sigma, 16-\sigma)$, the corresponding normalized eigenvector $\phi = (\phi_k)_{1\leq k\leq N} \in \mathbb{R}^N$ has the following property*

$$|\phi_N| > K\sqrt{\lambda}.$$

*Moreover, if $\phi^N \in \mathbb{R}^N$ is the eigenvector corresponding to the last eigenvalue $\lambda_N$, we have that*

$$\frac{|\phi_N^N|}{\sqrt{\lambda_N}} = O(h).$$

# Observability of the high-order eigenvectors

$$\phi^k = C_1 X_1^k + C_2 X_2^k + C_3 X_3^k + C_4 X_4^k$$

$$C_1 = \frac{\mathcal{C}}{X_1^{N+1} r_N^1}, \qquad C_2 = -\frac{\mathcal{C}}{X_2^{N+1} r_N^2}$$

$$C_3 = -\alpha\, C_1 \left(\frac{X_1}{X_3}\right)^{N+1} - \beta\, C_2 \left(\frac{X_2}{X_3}\right)^{N+1}$$

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$$\alpha = \frac{1}{2}\left(1 - i\frac{\sqrt{(2+\sqrt{\lambda})^2 - 4}}{\sqrt{4-(2-\sqrt{\lambda})^2}}\right), \qquad \beta = \frac{1}{2}\left(1 + i\frac{\sqrt{(2+\sqrt{\lambda})^2 - 4}}{\sqrt{4-(2-\sqrt{\lambda})^2}}\right)$$

$$r_N^j = \sqrt{\left(\left(\frac{X_4}{X_j}\right)^{N+1} - 1\right)\left(\left(\frac{X_3}{X_j}\right)^{N+1} - 1\right)} \qquad (j \in \{1,2\})$$

# Observability of the high-order eigenvectors

**Lemma**

*There exists $N_0 \in \mathbb{N}^*$ such that for each $N > N_0$ and any eigenvalue $\lambda$ of the matrix $A$ with the property that $\lambda \geq (3h \ln N)^4$ the following estimates hold:*

$$\frac{1}{X_1^{N+1}} = o(1)\sqrt{\lambda},$$

$$|1 - r_N^1| \leq \left(\frac{1}{X_1}\right)^{N+1},$$

$$r_N^2 \geq X_1^{N+1} - 1.$$

# Observability of the high-order eigenvectors

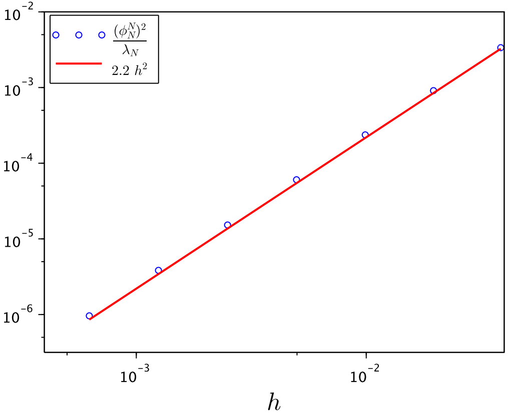


Figure: Evolution of the quantity $\dfrac{(\phi_N^N)^2}{\lambda_N}$ as a function of $h$.

# Characterisation of low eigenvalues and eigenvectors

**Proposition**

*Let* $\varepsilon \in (0,2)$*. There exist* $N_0 > 0$ *and* $d > 0$ *such that, for each* $N \geq N_0$*, the following estimate holds:*

$$\frac{1}{h^2}\left|\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n}\right| \geq dn \qquad \left(1 \leq n \leq N^{\frac{1}{6}(2-\varepsilon)}\right).$$

**Proposition**

*Let* $N \in \mathbb{N}^*$*,* $\sigma \in (0,1)$ *and* $\phi = (\phi_k)_{1\leq k\leq N}$ *be the normalized eigenvector of* $A$ *corresponding to the eigenvalue* $\lambda \in (0, 16-\sigma)$*. Then there exists a constant* $K > 0$*, independent of* $N$ *and* $\lambda$*, such that the following estimate holds*

$$|\phi_N| \geq K\sqrt{\lambda}.$$

# Low eigenvalues distribution

- Let $(\widetilde{A}, D(\widetilde{A}))$ be the operator in $L^2(0,1)$ associated to the clamped beam equation

$$\widetilde{A}\, u = \partial_x^4 u \qquad (u \in D(\widetilde{A})), \quad D(\widetilde{A}) = H^4(0,1) \cap H_0^2(0,1).$$

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- $\widetilde{A}$ has a sequence of simple eigenvalues $(\widetilde{\lambda}_n)_{n\geq 1}$:

$$\widetilde{\lambda}_n = \left(n + \frac{1}{2}\right)^4 \pi^4 + \upsilon_n \qquad (n \geq 1),$$

where $(\upsilon_n)_{n\geq 1}$ is a sequence converging exponentially to zero.

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  where $(\upsilon_n)_{n\geq 1}$ is a sequence converging exponentially to zero.
- Let $\varepsilon \in (0,2)$. There exist $N_0 > 0$ and $C > 0$ such that, for each $N \geq N_0$, the following estimate holds:

$$\left|\widetilde{\lambda}_n - \frac{\lambda_n}{h^4}\right| \leq C h^{\varepsilon} \qquad \left(1 \leq n \leq N^{\frac{1}{6}(2-\varepsilon)}\right).$$

# Low eigenvectors observability

- We employ a discrete multiplier method:

$$A\phi = \lambda\phi \quad | \quad \cdot J.D_{1c}\phi,$$

where

$$D_{1c} = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & \dots & 0 \\ -1 & 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 0 & 1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 0 & 1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 0 \end{pmatrix} \quad J = \begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ N-2 \\ N-1 \\ N \end{pmatrix}.$$

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- One deduce the following expression for $\phi_N$:

$$\phi_N^2 = \langle A\phi, \phi\rangle - \frac{\lambda}{4}\langle B\phi, \phi\rangle - \frac{h}{4}\left(4\phi_1^2 + 4\phi_N^2 - \phi_1\phi_2 - \phi_{N-1}\phi_N\right).$$

### Lemma

*With the above notation we have that*

1. $A = D_{1b}D_3 + M_1$,
2. $D_3 = D'_{1b}B + M_2$,
3. $B = D_{1b}D'_{1b} + M_3$,
4. $D'_{1b}(v.w) = D'_{1b}v.w + S'_0 v.D'_{1b}w$, *for every vectors* $v,\ w \in \mathbb{R}^N$, *where*
$$S_0 = \mathcal{I} - D_{1b}, \tag{1}$$
*where* $\mathcal{I}$ *denotes the identity matrix in* $\mathcal{M}_N(\mathbb{R})$.

$$M_1 = \begin{pmatrix} 4 & -1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & 2 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}.$$

## Proposition

*Let $T > 0$. There exist $N_0,\ n_T \in \mathbb{N}^*$ such that, for any $N \geq N_0$, the eigenvalues $\lambda_n$ of the matrix $A$ verify*

$$\sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \geq \frac{2\pi}{T} h^2 \qquad (n_T \leq n \leq N - n_T)\,. \tag{2}$$

Conclusion of the proof follows by:

- $\begin{pmatrix} Y_h(t) \\ \dot{Y}_h(t) \end{pmatrix} = \sum_{1\leq |n| \leq \gamma N} a_n e^{-i\,\mathsf{sgn}(n)\frac{\sqrt{\lambda_{|n|}}}{h^2}t} \Phi^n.$
- a Ingham's type inequality:

$$\sum_{1\leq |n|\leq \gamma N} |a_n|^2 \left| \frac{\phi_N^{|n|}}{\sqrt{\lambda_{|n|}}} \right|^2 \leq K' \int_0^T \left| \sum_{1\leq |n|\leq \gamma N} a_n e^{-i\,\mathsf{sgn}(n)\frac{\sqrt{\lambda_{|n|}}}{h^2}t} \frac{\phi_N^{|n|}}{\sqrt{\lambda_{|n|}}} \right|^2 dt.$$

# Numerical simulations

- We approach the discrete controls $v_h$ minimising the functional
$$J(v) = \int_0^T r(t)|v(t)|^2 dt$$
where $r \in C^\infty(0,T)$ is given by
$$r(t) = \begin{cases} 0 & (t \in (0, \frac{\alpha}{2}) \cup (T - \frac{\alpha}{2}, T)) \\ 1 & (t \in (\alpha, T - \alpha)). \end{cases}$$
- A classical conjugate gradient algorithm is used to minimise the dual functional $J^\star$.
- Newmark method is employed for the time discretization with a discretization step $\Delta t$ small enough.

# Numerical simulations

## A first example

$$u_0(x) = \sin^2(\pi x), \qquad u_1(x) = 0 \qquad (x \in (0,1))$$

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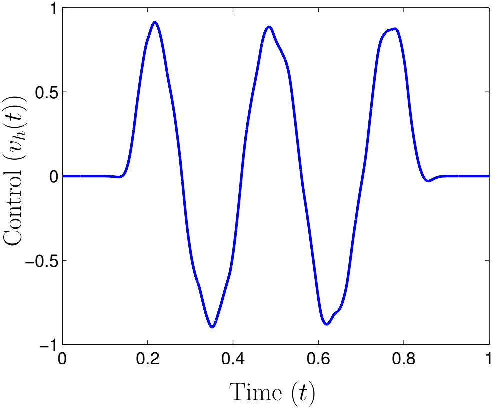



Figure: Control $v_h(t)$

$$u_0(x) = \sin^2(\pi x), \qquad u_1(x) = 0 \qquad (x \in (0,1))$$

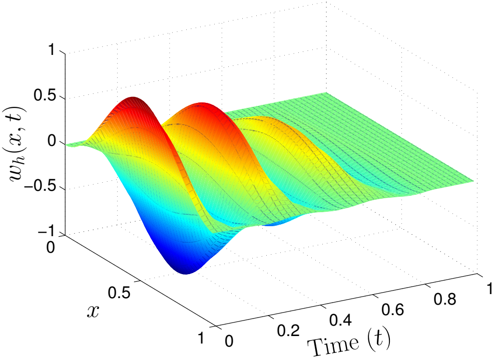



Figure: Control solution.

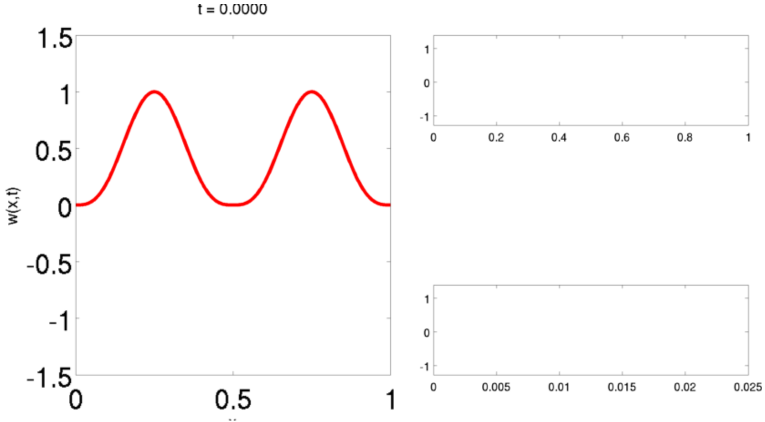
# Numerical simulations

## A more oscillating example

$$u_0(x) = \sin^2(2\pi x), \qquad u_1(x) = 0 \qquad (x \in (0,1))$$

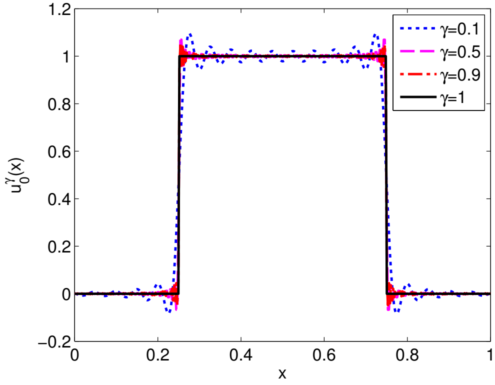
# Numerical simulations

## A more oscillating example



$$u_0(x) = \mathbb{1}_{(\frac{1}{4},\frac{3}{4})}(x), \qquad u_1(x) = 0 \qquad (x \in (0,1)).$$

$$u_0^\gamma = \sum_{n=1}^{[\gamma N]} \langle u_0, \phi^n \rangle_0 \phi^n \in \mathbb{C}^N.$$

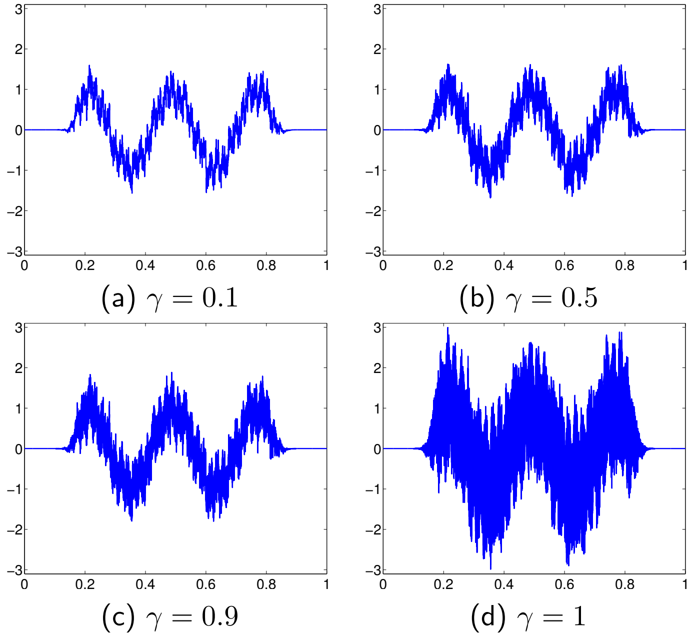


# Numerical simulations

## Number of iterations needed for the CG to converge

| | $\gamma = 0.1$ | $\gamma = 0.5$ | $\gamma = 0.9$ | $\gamma = 1$ |
|---|---|---|---|---|
| $N = 25$ | 4 | 6 | 12 | 29 |
| $N = 50$ | 4 | 6 | 15 | 52 |
| $N = 100$ | 4 | 6 | 17 | 87 |
| $N = 200$ | 4 | 6 | 20 | 168 |
| $N = 400$ | 4 | 6 | 19 | 321 |

Table: Number of iterations needed for the convergence of the conjugate gradient algorithm for initial data $(u_0^\gamma, 0)$ and different values of $N$.




(a) $\gamma = 0.1$ (b) $\gamma = 0.5$

(c) $\gamma = 0.9$ (d) $\gamma = 1$

Figure: Controls obtained for $N = 400$ and different values of $\gamma$.

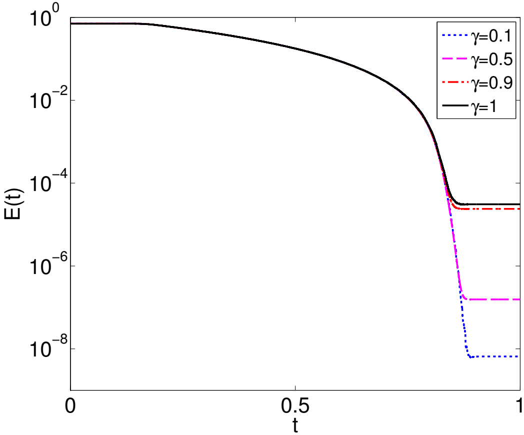



Figure: Energy of controlled solutions corresponding to $u_0^\gamma$ for different values of $\gamma$ and $N = 400$.

# Conclusion and perspectives

**Conclusion:**

- We proved that the observability inequality associated to a finite-differences semi-discretization of the clamped beam equation holds uniformly for filtered initial data;
- The filtration threshold is sharp.
- A precise analysis of the spectral properties of the discrete operator was needed.

**Perspectives:**

- two-dimensional case?
- other less academic numerical schemes?

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# Thank you for the attention!