On the stabilization of the incompressible Navier-Stokes equations in a 2d channel with a normal control

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Outline

Outline

Incompressible Navier-Stokes equations in a 2-d channel:

$$\Omega = \mathbb{T} \times (0,1), \text{ where } \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p = 0, & \text{in } (0,\infty) \times \Omega, \\ \operatorname{div} u = 0, & \text{in } (0,\infty) \times \Omega, \\ u(t, x_1, 0) = (0,0), & \text{on } (0,\infty) \times \mathbb{T}, \\ u(t, x_1, 1) = (0, v(t, x_1)), & \text{on } (0,\infty) \times \mathbb{T}, \\ u(0, x_1, x_2) = u^0(x_1, x_2), & \text{in } \Omega. \end{cases}$$

- $u = u(t, x_1, x_2) \in \mathbb{R}^2$ is the velocity.
- $p = p(t, x_1, x_2)$ is the pressure.
- $v = v(t, x_1)$ is the control function, acting on the normal component only.

Choose v to stabilize the state u.

Motivation and related topics

Motivation: Controllability/Stabilization of fluid-structure models with controls acting on the structure.
See Lions Zuazua '95, Osses Puel '99, '09, Lequeurre '13, ...

Related topics:

- Controllability of incompressible Navier-Stokes equations....
  Fursikov Imanuvilov '96, Fernandez-Cara Guerrero Imanuvilov Puel '04, ...
- ... with controls having zero components:
  Coron Guerrero '09, Carreno Guerrero '13, Coron Lissy '15,...
- Coupled parabolic systems with one boundary control:
  Ammar-Khodja Benabdallah Gonzalez-Burgos de Teresa '11, Duprez Lissy '15...
- Stabilization for incompressible Navier-Stokes equations:
  Krstic et al '01, Raymond '06, Barbu '07, Triggiani '07, Vazquez Coron Trélat '08, Munteanu '12,...

To be more precise....

Our goal

Get a local stabilization result around the state $(u, p) = (0, 0)$.

Linearized equations:

$$\begin{cases} \partial_t u - \Delta u + \nabla p = 0, & \text{in } (0, \infty) \times \Omega, \\ \text{div } u = 0, & \text{in } (0, \infty) \times \Omega, \\ u(t, x_1, 0) = (0, 0), & \text{on } (0, \infty) \times \mathbb{T}, \\ u(t, x_1, 1) = (0, v(t, x_1)), & \text{on } (0, \infty) \times \mathbb{T}, \\ u(0, x_1, x_2) = u^0(x_1, x_2), & \text{in } \Omega, \end{cases}$$

??? $\rightsquigarrow$ *The linearized equations are already stable!* Taking $v = 0$,

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |u(t,x)|^2\, dx\right) + \int_\Omega |\nabla u(t,x)|^2\, dx = 0.$$

$\rightsquigarrow$ Exponential decay like $t \mapsto \exp(-\pi^2 t)$!
(also true for the non-linear model).

To be more precise.... Part 2

Get a local stabilization result around the state $(u, p) = (0, 0)$
At an exponential rate larger than π^2.

Linearized equations:

$$\begin{cases} \partial_t u - \Delta u + \nabla p = 0, & \text{in } (0, \infty) \times \Omega, \\ \operatorname{div} u = 0, & \text{in } (0, \infty) \times \Omega, \\ u(t, x_1, 0) = (0, 0), & \text{on } (0, \infty) \times \mathbb{T}, \\ u(t, x_1, 1) = (0, v(t, x_1)), & \text{on } (0, \infty) \times \mathbb{T}, \\ u(0, x_1, x_2) = u^0(x_1, x_2), & \text{in } \Omega, \end{cases}$$

Difficulty:

$$\operatorname{div} u = 0 \text{ in } (0, \infty) \times \Omega \Rightarrow \int_{\mathbb{T}} v(t, x_1)\, dx_1 = 0 \text{ for all } t > 0.$$

To be more precise.... Part 3

The 0-mode:

$$u_0(t, x_2) = \int_{\mathbb{T}} u(t, x_1, x_2)\, dx_1$$

satisfies the uncontrolled heat equation

$$\begin{cases} \partial_t u_{0,1} - \partial_{22} u_{0,1} = 0, & \text{in } (0,\infty) \times (0,1), \\ u_{0,1}(t,0) = u_{0,1}(t,1) = 0, & \text{on } (0,\infty), \\ u_{0,2}(t, x_2) = 0, & \text{in } (0,\infty) \times (0,1). \end{cases}$$

Consequence

The solutions of the linearized equations decay like $\exp(-\pi^2 t)$ and, considering

$$u(t,x) = e^{-\pi^2 t} \Psi_0(x_2) \text{ with } \Psi_0 = \Psi_0(x_2) = \sqrt{\frac{2}{\pi}} \begin{pmatrix} \sin(\pi x_2) \\ 0 \end{pmatrix},$$

this decay estimate is sharp whatever the control v is.

Main result

Theorem (S. Chowdhury, S.E., J.-P. Raymond 2016)

Let $\omega_0 > 0$ be such that $0 < \omega_0 < 4\pi^2$.
There exists $\gamma > 0$ such that for all $u_0 \in \mathbf{V}^1_0(\Omega)$ with $\|u_0\|_{\mathbf{V}^1_0(\Omega)} \leq \gamma$, there exists $v \in L^2((0,\infty) \times \mathbb{T})$ satisfying $\displaystyle\int_{\mathbb{T}} v(t, x_1)\, dx_1 = 0$ for all $t > 0$ such that the solution (u, p) of the incompressible Navier-Stokes equation satisfies, for some constant $C > 0$ independent of t,

$$\forall t \geq 0, \quad \|u(t)\|_{\mathbf{V}^1(\Omega)} \leq Ce^{-\omega_0 t}.$$

$$\mathbf{V}^1(\Omega) = \left\{u = (u_1, u_2) \in H^1(\Omega) \times H^1(\Omega) \,|\, \mathrm{div}\ u = 0\right\},$$
$$\mathbf{V}^1_0(\Omega) = \left\{u \in \mathbf{V}^1(\Omega) \,|\, u(x_1, 0) = u(x_1, 1) = 0 \text{ for } x_1 \in \mathbb{T}\right\}.$$

Comments

- Straightforward when $\omega < \pi^2$
  $\rightsquigarrow$ Interesting case $\omega \in (\pi^2, 4\pi^2)$.
- $4\pi^2$ is the second eigenvalue of the elliptic operator generating the heat equation satisfied by the 0-mode:

$$\begin{cases} \partial_t u_{0,1} - \partial_{22} u_{0,1} = 0, & \text{in } (0,\infty)\times(0,1), \\ u_{0,1}(t,0) = u_{0,1}(t,1) = 0, & \text{on } (0,\infty), \\ u_{0,2}(t,x_2) = 0, & \text{in } (0,\infty)\times(0,1). \end{cases}$$

- The stabilization result cannot be true for the linearized model
  $\Rightarrow$ We have to use the non-linearity to improve the exponential decay.
  Strategy based on the so-called Power Series Expansion: see Coron Crépeau '04, Cerpa '07, Cerpa Crépeau '09, Coron Rivas '15.

Outline

Strategy Part 1

Write $u = \varepsilon\alpha + \varepsilon^2\beta$, $v = \varepsilon v_1 + \varepsilon^2 v_2$, with

$$\begin{cases} \partial_t\alpha - \Delta\alpha + \nabla p_1 = 0, & \text{in } (0,\infty)\times\Omega, \\ \operatorname{div}\,\alpha = 0, & \text{in } (0,\infty)\times\Omega, \\ \alpha(t,x_1,0) = (0,0), & \text{on } (0,\infty)\times\mathbb{T}, \\ \alpha(t,x_1,1) = (0, v_1(t,x_1)), & \text{on } (0,\infty)\times\mathbb{T}, \\ \alpha(0,x_1,x_2) = \alpha^0(x_1,x_2), & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \partial_t\beta - \Delta\beta + \nabla p_2 = -(\alpha+\varepsilon\beta)\cdot\nabla(\alpha+\varepsilon\beta), & \text{in } (0,\infty)\times\Omega, \\ \operatorname{div}\,\beta = 0, & \text{in } (0,\infty)\times\Omega, \\ \beta(t,x_1,0) = (0,0), & \text{on } (0,\infty)\times\mathbb{T}, \\ \beta(t,x_1,1) = (0, v_2(t,x_1)), & \text{on } (0,\infty)\times\mathbb{T}, \\ \beta(0,x_1,x_2) = \beta^0(x_1,x_2), & \text{in } \Omega, \end{cases}$$

Strategy

Part 1

Write $u = \varepsilon \alpha + \varepsilon^2 \beta$, $v = \varepsilon v_1 + \varepsilon^2 v_2$, with

$$\begin{cases} \partial_t \alpha - \Delta \alpha + \nabla p_1 = 0, & \text{in } (0,\infty) \times \Omega, \\ \text{div } \alpha = 0, & \text{in } (0,\infty) \times \Omega, \\ \alpha(t, x_1, 0) = (0,0), & \text{on } (0,\infty) \times \mathbb{T}, \\ \alpha(t, x_1, 1) = (0, v_1(t, x_1)), & \text{on } (0,\infty) \times \mathbb{T}, \\ \alpha(0, x_1, x_2) = \alpha^0(x_1, x_2), & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \partial_t \beta - \Delta \beta + \nabla p_2 = -\alpha \cdot \nabla \alpha, & \text{in } (0,\infty) \times \Omega, \\ \text{div } \beta = 0, & \text{in } (0,\infty) \times \Omega, \\ \beta(t, x_1, 0) = (0,0), & \text{on } (0,\infty) \times \mathbb{T}, \\ \beta(t, x_1, 1) = (0, v_2(t, x_1)), & \text{on } (0,\infty) \times \mathbb{T}, \\ \beta(0, x_1, x_2) = \beta^0(x_1, x_2), & \text{in } \Omega, \end{cases}$$

Strategy

- α satisfies the linearized incompressible Navier-Stokes equations.

$\Rightarrow$ If α contains 0-modes decaying slower than $\exp(-\omega_0 t)$, one cannot achieve an exponential decay rate ω_0.

$\Rightarrow$ The component of the solution u on the eigenfunction

$$\Psi_0 = \Psi_0(x_2) = \sqrt{\frac{2}{\pi}} \begin{pmatrix} \sin(\pi x_2) \\ 0 \end{pmatrix}$$

- Is in β.
- Should be handled by constructing a suitable α.

Preliminaries

• The Stokes operator A is self-adjoint, positive definite, with compact resolvent on the space
$\mathbf{V}_n^0(\Omega) = \left\{ u \in (L^2(\Omega))^2 \mid \mathrm{div}\,(u) = 0 \text{ on } \Omega \text{ and } u \cdot n = 0 \text{ on } \Gamma \right\}$
$\Rightarrow$ Sequences of positive eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ and corresponding orthonormal basis of eigenvectors (Ψ_j).

$$A\Psi = \lambda\Psi \;\Leftrightarrow \begin{cases} -\Delta\Psi + \nabla q = \lambda\Psi, & \text{in } \Omega, \\ \mathrm{div}\ \Psi = 0, & \text{in } \Omega, \\ \Psi = 0, & \text{on } \Gamma, \end{cases}$$

Adjoint of the control operator: $B^*\Psi = q(x_1, 1) - \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{T}} q(x_1, 1) dx_1.$

Lemma

$A\Psi = \lambda\Psi$ and $B^*\Psi = 0$ imply $\Psi(x) = \Psi(x_2)$.

Decomposition of the space $\mathbf{V}_n^0(\Omega)$:

- A stable space: $\mathbf{Z}_s = \text{Span}\,\{\Phi \,|\, A\Phi = \lambda\Phi, \text{ with } \lambda > \omega\}$.
- An unstable space: $\mathbf{Z}_u = \mathbf{Z}_s^\perp$, itself decomposed as
  - An unstable uncontrollable space $\mathbf{Z}_{uu} = \text{Span}\,\Psi_0$.
  - An unstable detectable space $\mathbf{Z}_{ud} = \mathbf{Z}_{uu}^\perp \cap \mathbf{Z}_u$.

Corresponding (orthogonal) projections: $\mathbb{P}_s$, $\mathbb{P}_u$, $\mathbb{P}_{ud}$ and $\mathbb{P}_{uu}$.

Strategy

Part 3

Iterative strategy: $(0, \infty) = \cup_{n \in \mathbb{N}} [nT, (n+1)T]$ for some $T > 0$.
$(T = 1)$.

Starting point: $u^0 = \varepsilon \alpha^0 + \varepsilon^2 \beta^0$ with

$$\|\alpha^0\|^2_{\mathbf{V}^1_0(\Omega)} + \|\beta^0\|_{\mathbf{V}^1_0(\Omega)} \leq 1, \quad \text{and} \quad \mathbb{P}_{uu}\alpha^0 = 0.$$

Initialization Step: On $[0, T]$, choose

- v_1 such that $\mathbb{P}_u \alpha(T) = 0$.
- $v_2 = 0$.

Strategy

Part 4

Iteration step: In each time interval $[nT, (n+1)T]$, we design controls v_1 and v_2 such that

$$\mathbb{P}_u\alpha((n+1)T) = 0, \quad \text{and} \quad \mathbb{P}_u\beta((n+1)T) = 0,$$

where β is the solution of

$$\begin{cases} \partial_t\beta - \Delta\beta + \nabla p = -\alpha\cdot\nabla\alpha, & \text{in } (nT,(n+1)T)\times\Omega, \\ \operatorname{div}\,\beta = 0, & \text{in } (nT,(n+1)T)\times\Omega, \\ \beta(t,x_1,0) = (0,0), & \text{on } (nT,(n+1)T)\times\mathbb{T}, \\ \beta(t,x_1,1) = (0,v_2(t,x_1)), & \text{on } (nT,(n+1)T)\times\mathbb{T}, \\ \beta(nT^+,x) = \beta(nT^-,x), & \text{in } \Omega. \end{cases}$$

Key Lemma

There exist control functions $v^a, v^b \in H^1_0(0,T; H^2(\mathbb{T}) \cap L^2_0(\mathbb{T}))$ such that, for all $a, b \in \mathbb{R}$, the solution α of

$$\begin{cases} \partial_t \alpha - \Delta \alpha + \nabla p_1 = 0, & \text{in } (0,T) \times \Omega, \\ \operatorname{div} \alpha = 0, & \text{in } (0,T) \times \Omega, \\ \alpha(t, x_1, 0) = (0,0), & \text{on } (0,T) \times \mathbb{T}, \\ \alpha(t, x_1, 1) = (0, (av^a + bv^b)(t, x_1)), & \text{on } (0,T) \times \mathbb{T}, \\ \alpha(0,x) = 0, & \text{in } \Omega, \end{cases}$$

satisfies $\alpha(T) = 0$ in Ω, and such that the solution β of

$$\begin{cases} \partial_t \beta - \Delta \beta + \nabla p_2 = -\alpha \cdot \nabla \alpha, & \text{in } (0,T) \times \Omega, \\ \operatorname{div} \beta = 0, & \text{in } (0,T) \times \Omega, \\ \beta(t, x_1, 0) = \beta(t, x_1, 1) = (0,0), & \text{on } (0,\infty) \times \mathbb{T}, \\ \beta(0,x) = 0, & \text{in } \Omega, \end{cases}$$

satisfies $\mathbb{P}_{uu} \beta(T) = ab \Psi_0$.

Difficulties

- Generating many trajectories for α starting from 0 and ending at 0.
  $\rightsquigarrow$ Null-controllability results on the Eq. of the 1st mode.

- Generate trajectories such that $\alpha \cdot \nabla \alpha$ allows for β to enter in the missing direction.
  - Specific solutions with separated variables
  - Contradiction argument.

Ideas of the proof: Generation of trajectories

- Take

$$v^a(t,x_1) = v^c(t)\cos(x_1), \quad v^b(t,x_1) = v^s(t)\sin(x_1).$$

$\rightsquigarrow$ α^a and α^b are supported on the first mode of the equations:

$$\alpha^a(t,x_1,x_2) = \begin{pmatrix} \sin(x_1)\alpha_1^c(t,x_2) \\ \cos(x_1)\alpha_2^c(t,x_2) \end{pmatrix}$$

- The equation satisfied by the first modes of the linear incompressible Stokes equations is null-controllable.

$\rightsquigarrow$ Proof by spectral estimates.

$\longrightarrow$ We can generate many trajectories α going from 0 to 0.

The projection on Ψ_0 of the corresponding $\beta(T)$

$$e^{\nu\pi^2 T}\langle \beta(T), \Psi_0\rangle = \pi^{5/2}\int_0^T v^s(t) q(t,1)\, dt,$$

where q is obtained by solving

$$\begin{cases} -\partial_t Z + Z - \partial_{22} Z + \begin{pmatrix} q \\ \partial_2 q \end{pmatrix} = F(t, x_2), & \text{in } (0,T)\times(0,1), \\ -Z_1 + \partial_2 Z_2 = 0, & \text{in } (0,T)\times(0,1), \\ Z(t,0) = Z(t,1) = (0,0), & \text{in } (0,T), \\ Z(T, x_2) = 0, & \text{in } (0,1). \end{cases}$$

with $F(t,x_2) = \cos(\pi x_2) e^{\pi^2 t}\begin{pmatrix} \alpha_2^c(t,x_2) \\ \alpha_1^s(t,x_2) \end{pmatrix}$, depending only on $v^a(t,x_1) = v^c(t)\cos(x_1)$.

$\rightsquigarrow$ Show the existence of v^a/v^c such that $\|q(t,1)\|_{L^2(0,T)} \neq 0$.

Construction of v^c, generation of a suitable trajectory

For $\mu \in \mathbb{R}$, introduce $(\alpha^*(x_2), p^*(x_2))$ solving

$$\begin{cases} \mu\alpha_1^* + \alpha_1^* - \partial_{22}\alpha_1^* - p^* = 0, & \text{in } (0,1), \\ \mu\alpha_2^* + \alpha_2^* - \partial_{22}\alpha_2^* + \partial_2 p^* = 0, & \text{in } (0,1), \\ \alpha_1^* + \partial_2\alpha_2^* = 0, & \text{in } (0,1), \\ \alpha_1^*(0) = \alpha_1^*(1) = \alpha_2^*(0) = 0, \quad \alpha_2^*(1) = 1. & \end{cases}$$

Then $\overline{\alpha}(t, x_1, x_2) = e^{\mu t}(\sin(x_1)\alpha_1^*(x_2), \cos(x_1)\alpha_2^*(x_2))$, $\overline{v}(t) = e^{\mu t}$, solves the linear Stokes equations.

Lemma

There exists a suitable $\mu \in \mathbb{R}$ such that if $\alpha(t) = \overline{\alpha}(t)$ on some time interval then the boundary pressure $q(t, 1)$ given by the aforementioned process cannot be identically 0 on that time interval.

Reduction to the stationary case and numerically checked.

Construction of v^a/v^c and v^b/v^s

Construction of v^a/v^c in 4 steps:

- On $(0, T/4)$, control α^a to go from 0 to $\overline{\alpha}(T/4)$.
- On $(T/4, T/2)$, take $v^a(t) = e^{\mu t}$ and $\alpha^a(t) = \overline{\alpha}(t)$.
  hence $\|q(t)\|_{L^2(T/4,T/2)} \neq 0$.
- On $(T/2, 3T/4)$, control α^a goes from $\overline{\alpha}(T/2)$ to 0.
- On $(3T/4, T)$, take $v^a(t) = 0$, and $\alpha^a(t) = 0$,
  hence $q(t) = 0$ on $(3T/4, T)$.

Construction of v^b/v^s:

- On $(0, 3T/4)$, take v^s such that $\displaystyle\int_0^{3T/4} v^s(t) q(t,1)\, dt = 1$.
- On $(3T/4, T)$, control α^b to go from $\alpha^b(3T/4)$ to 0.

Outline

Open Question

- Exponential stabilization result at a rate higher than $4\pi^2$?

Difficulty: One has to guarantee that we can enter the space of missing directions

$$\text{Span}\left\{\begin{pmatrix}\sin(\pi x_2)\\0\end{pmatrix},\begin{pmatrix}\sin(2\pi x_2)\\0\end{pmatrix}\right\}$$

in both directions independently.

This is OK !

But exponential stabilization at any given rate is open (even if probably true with our techniques...), so is the controllability of the system.

Thank you for your attention!

Comments Welcome

Reference:
Open loop stabilization of incompressible Navier-Stokes equations in a 2d channel using power series expansion.
S. Chowdhury, S. Ervedoza, and J.-P. Raymond, in preparation.