A result on the boundary stabilization of systems of conservation laws in the context of weak entropy solutions

Olivier Glass
(based on a paper with J.-M. Coron, S. Ervedoza, S. Goshal and V. Perrollaz)

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Introduction

- We discuss control problems of one-dimensional hyperbolic systems of conservation laws:

$$u_t + f(u)_x = 0, \quad f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n, \tag{SCL}$$

  satisfying the (strict) hyperbolicity condition that at each point

$$df \text{ has } n \text{ distinct real eigenvalues } \lambda_1 < \cdots < \lambda_n.$$

- Typically: compressible fluid flows, fluid through a canal, traffic flow, etc.

Characteristic fields

- Corresponding to the characteristic speeds $\lambda_1 < \cdots < \lambda_n$, the Jacobian $A(u) := df(u)$ has n right eigenvectors $r_i(u)$.
- We denote $(\ell_i)_{i=1,\ldots,n}$ the left eigenvectors of $df(u)$ satisfying $\ell_i \cdot r_j = \delta_{ij}$.
- The characteristic families will be supposed to be genuinely non-linear (GNL), that is:

$$\nabla\lambda_i \cdot r_i \neq 0 \quad \text{for all } u \text{ in } \Omega.$$

$\Rightarrow$ we normalize GNL fields as to satisfy $\nabla\lambda_i \cdot r_i = 1$.

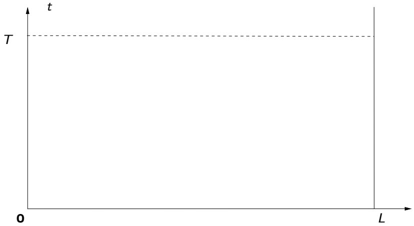
Controllability problem

- Domain: $[0, T] \times [0, L]$.
- State of the system $u(t, \cdot) \in BV(0, L)$
- Control: the "boundary data" on one or both sides. When one controls on one side only, the boundary condition on the other side is fixed.
- Exact controllability: given u_0 and u_1, can we find a boundary control for the system driving u_0 to u_1?
- Controllability to constant states: given u_0 and given $\overline{u}_1$ a constant state, can we find a boundary control for the system driving u_0 to $\overline{u}_1$?

Reformulation of the controllability problem

One can reformulate the controllability problem as follows.

- Exact controllability: given u_0 and u_1, can we find a solution of the system driving u_0 to u_1?
- Controllability to constant states: given u_0 and given $\overline{u}_1$ a constant state, can we find a solution of the system driving u_0 to $\overline{u}_1$?



Stabilization problem

- For this problem, we will suppose moreover that the characteristic speeds are stricly separated from 0:

$$\lambda_1 < \cdots < \lambda_m < 0 < \lambda_{m+1} < \cdots < \lambda_n.$$

- We will be interested in boundary conditions put in the following form:

$$\begin{pmatrix} u_+(t,0) \\ u_-(t,L) \end{pmatrix} = G \begin{pmatrix} u_+(t,L) \\ u_-(t,0) \end{pmatrix}$$

  with

$$u_+ := (u_{m+1}, \ldots, u_n) \text{ and } u_- := (u_1, \ldots, u_m).$$

- We consider an equilibrium point $\overline{u}$ of the system. To simplify, we fix $\overline{u} = 0$.
- The question is to design G so that $\overline{u}$ becomes an asymptotically stable point for the resulting closed-loop system.

Stabilization problem

- We recall that a point $\overline{u}$ is called stable when for any neighborhood $\mathcal{V}$ of $\overline{u}$, there exists a neighborhood $\mathcal{U}$ of $\overline{u}$ such that any trajectory of the system starting from $\overline{u}$ stays in $\mathcal{V}$ for all $t \geq 0$.

- It is called asymptotically stable when moreover any trajectory starting from $\mathcal{U}$ satisfies $u(t, \cdot) \to \overline{u}$ as $t \to +\infty$.

- It is called exponentially stable when any trajectory starting from some neighborhood $\mathcal{U}$ of $\overline{u}$ satisfies

$$\|u(t, \cdot)\| \leq C \exp(-\gamma t) \|u(0, \cdot)\| \quad \text{for all } t \geq 0,$$

for some fixed $\gamma > 0$ and $C > 0$.

Systems of conservation laws and gradient catastrophe

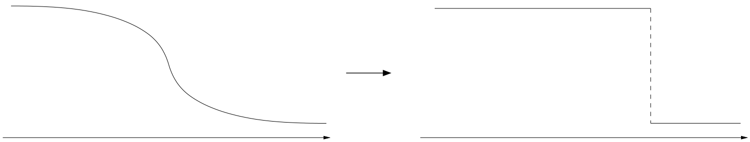
- Nonlinear hyperbolic systems of conservation laws

$$U_t + f(U)_x = 0, \quad f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$$

develop in general singularities in finite time.

- This easy to see for instance for the Burgers equation:

$$u_t + (u^2)_x = 0.$$



Two types of solutions

- One can either work with regular solutions with small C^1-norm (for some time – semi-global solutions), or with discontinuous (weak) solutions.
- Weak solutions can account for shock waves.
- When working with weak solutions it is natural for the sake of uniqueness to consider weak solutions which satisfy entropy conditions.
- This is the framework in which we work here: entropy solutions.
- More precisely, the solutions will be of bounded variation, with small total variation in x ("à la Glimm").

Entropy conditions

Definition

An entropy/entropy flux couple for a hyperbolic system of conservation laws (SCL) is defined as a couple of regular functions $(\eta, q) : \Omega \to \mathbb{R}$ satisfying:

$$\forall u \in \Omega, \quad D\eta(u) \cdot Df(u) = Dq(u).$$

Definition

A function $u \in L^\infty(0, T; BV(0, L)) \cap \mathcal{L}ip(0, T; L^1(0, L))$ is called an entropy solution of (SCL) when, for any entropy/entropy flux couple (η, q), with η convex, one has in the sense of measures

$$\eta(u)_t + q(u)_x \leq 0,$$

that is, for all $\varphi \in \mathcal{D}((0, T) \times (0, L))$ with $\varphi \geq 0$,

$$\int_{(0,T)\times(0,L)} \big(\eta(u(t,x))\varphi_t(t,x) + q(u(t,x))\varphi_x(t,x)\big)\, dx\, dt \geq 0.$$

Entropy conditions, 2

- Of course $(\eta, q) = (\pm \mathrm{Id}, \pm f)$ are entropy/entropy flux couples. So entropy solutions are particular cases of weak solutions.

- The entropy inequalities are automatically satisfied by vanishing viscosity limits:

$$u^\varepsilon \to u \text{ with } \; u^\varepsilon_t + f(u^\varepsilon)_x - \varepsilon u^\varepsilon_{xx} = 0.$$

- Glimm (1965) showed the existence of global entropy solutions with the assumption of small total variation, that is when $\partial_x u_0$ is small in the space of bounded measures.

Previous results in the classical case (controllability)

- Theorem (Li-Rao, 2002): Consider

$$\partial_t u + A(u)u_x = F(u),$$

such that $A(u)$ has n distinct real eigenvalues such that

$$\lambda_1(u) < \cdots < \lambda_k(u) \leq -c < 0 < c \leq \lambda_{k+1}(u) < \cdots < \lambda_n(u),$$

and

$$F(0) = 0.$$

Then for all $\phi, \psi \in C^1([0,1])$ such that $\|\phi\|_{C^1} + \|\psi\|_{C^1} < \varepsilon$, there exists a solution $u \in C^1([0,T] \times [0,1])$ such that

$$u_{|t=0} = \phi, \text{ and } u_{|t=T} = \psi.$$

- Many works since in this context of semi-global solutions! See the book of Li Ta-Tsien on the subject.

Stabilization problem

- One would like to extend to the realm of BV entropy solutions stabilization results that were obtained in the context of classical solutions, such as
  - Slemrod, Greenberg-Li, ...
  - Bastin-Coron, Bastin-Coron-d'Andrea-Novel, Bastin-Coron-d'Andrea-Novel-de Halleux-Prieur, Bastin-Coron-Krstic-Vazquez, ...
  - Leugering-Schmidt, Dick-Gugat-Leugering, Gugat-Herty,...
  - Ta-Tsien Li, Tie Hu Qin, ...
  - Many others!

Some results in the case of entropy solutions

- Several works on the scalar case:
  - Ancona and Marson (1998),
  - Horsin (1998),
  - Perrollaz (2011),
  - Adimurthi-Gowda-Goshal (2013),
  - Andreianov-Donadello-Marson (2015),
  - Adimurthi-Goshal-Marcati (2016),
- Several works on the system case:
  - Bressan-Coclite (asymptotic result and a counterexample, 2002),
  - Ancona-Coclite (Temple systems, 2002),
  - Ancona-Marson (one-side open loop stabilization, 2007),
  - G. (2007, 2014),
  - Andreianov-Donadello-Ghoshal-Razafison (2015, triangular system),
  - T. Li-L. Yu (2015, partially LD systems),
  - Coron-Ervedoza-G.-Goshal-Perrollaz (2015).

A simple framework

- Here we consider 2×2 systems of conservation laws:

  $$u_t + f(u)_x = 0 \quad \text{in} \quad [0, +\infty) \times [0, L],$$

  with characteristic speeds $\lambda_1 < \lambda_2$ and satisfying the conditions:

  - each characteristic field is GNL,
  - velocities are positive: $0 < \lambda_1 < \lambda_2$.

- The boundary conditions are as follows:

  $$u(t, 0) = Ku(t, L),$$

  where K is a 2×2 (real) matrix.

- The goal is to find conditions on K such ensuring the (exponential) stability of the system.

Main result

Theorem (Coron-Ervedoza-G.-Goshal-Perrollaz)

Suppose the above assumptions satisfied. If the matrix K satisfies

$$\inf_{\alpha \in (0,+\infty)} \Big(\max \big\{ |\ell_1(0) \cdot K r_1(0)| + \alpha |\ell_2(0) \cdot K r_1(0)|, \\ \alpha^{-1} |\ell_1(0) \cdot K r_2(0)| + |\ell_2(0) \cdot K r_2(0)| \big\} \Big) < 1,$$

then there exist positive constants C, ν, $\varepsilon_0 > 0$, such that for every $u_0 \in BV(0,L)$ satisfying

$$|u_0|_{BV} \leq \varepsilon_0,$$

there exists an entropy solution u of the system in $L^\infty(0,\infty; BV(0,L))$ satisfying $u(0,\cdot) = u_0(\cdot)$ and the boundary conditions for almost all times, such that

$$|u(t)|_{BV} \leq C \exp(-\nu t) |u_0|_{BV}, \qquad t \geq 0.$$

- See Sablé-Tougeron for a very general related result on global existence of solutions on an interval with local feedbacks.

Rewriting the condition

Denoting for $p \in [1,\infty)$

$$\|(x_1,\dots,x_n)\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad \|(x_1,\dots,x_n)\|_\infty := \max_{i=1\dots n} |x_i|$$

$$\|M\|_p := \max_{\|x\|_p=1} \|Mx\|_p \ \text{ for } \ M \in R^{n\times n},$$

one defines

$$\rho_p(K) := \inf\{\|\Delta K \Delta^{-1}\|_p,\ \Delta \text{ diagonal with positive entries}\}.$$

It is easy to check that

$$\inf_{\alpha\in(0,+\infty)} \big(\max\big\{|\ell_1(0)\cdot Kr_1(0)| + \alpha|\ell_2(0)\cdot Kr_1(0)|, \\ \alpha^{-1}|\ell_1(0)\cdot Kr_2(0)| + |\ell_2(0)\cdot Kr_2(0)|\big\}\big) = \rho_1(K),$$

so that the condition can be written as $\rho_1(K) < 1$.

Analogous conditions

- For the same question for classical solutions in C^m-norm ($m \geq 1$), a sufficient condition is:
$$\rho_\infty(K) < 1.$$
Cf. T. H. Qin, Y. C. Zhao, T. Li and Bastin-Coron.

- In the case of Sobolev spaces $W^{m,p}([0, L])$ with $m \geq 2$ and $p \in [1, +\infty]$, a sufficient condition is:
$$\rho_p(K) < 1.$$
Cf. Coron-d'Andréa-Novel-Bastin for $p = 2$, Coron-Nguyen for general p.

Remarks

- One can actually show that

$$\rho_1(K) = \rho_\infty(K).$$

- The known results on the existence of a standard Riemann semigroup for initial-boundary problem do not seem to cover our situation exactly and uniqueness of solutions in the spirit of Bressan-LeFloch or Bressan-Goatin seems open.

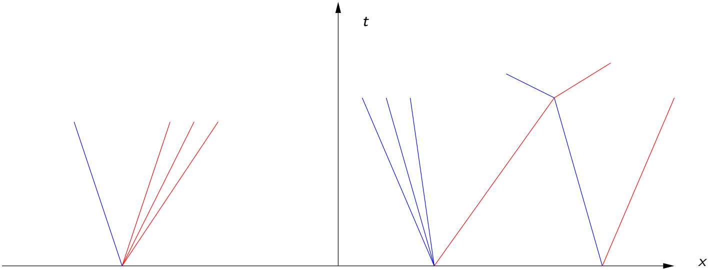
  Cf. Amadori, Amadori-Colombo, Colombo-Guerra, Donadello-Marson, Sablé-Tougeron,. . .

A general idea

- One constructs solutions using the wave-front tracking approach (here, DiPerna's approach since we consider 2×2 systems)
- Then the result relies on a Lyapunov function.
- This Lyapunov function is mainly inspired by two sources:
  - Lyapunov functions constructed in the classical case, cf. Coron-Bastin-d'Andrea-Novel, Coron-Bastin, ...
  - Glimm's functional used to construct entropy solutions in BV

1. Wave-front tracking algorithm

- Solutions are constructed directly using a wave-front tracking approach (cf. Dafermos, DiPerna, Bressan, ...):
  - one constructs a sequence of approximations of a solutions,
  - these approximations are piecewise constant functions on $\mathbb{R}_+ \times \mathbb{R}$ where the discontinuities are straight lines separating states connected by shocks or rarefactions,



The Riemann problem

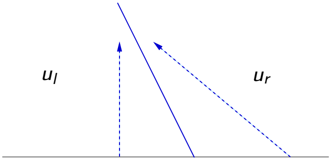
- Find autosimilar solutions $u = \overline{u}(x/t)$ to

$$\begin{cases} u_t + (f(u))_x = 0 \\ u_{|\mathbb{R}^-} = u_l \text{ and } u_{|\mathbb{R}^+} = u_r. \end{cases}$$

- Solved by introducing Lax's curves which consist of points that can be joined starting from u_l (in the case of GNL fields):
  - either by a shock,
  - or by a rarefaction wave.

Shocks and rarefaction waves (GNL fields)

Shocks



Discontinuities satisfying:

- Rankine-Hugoniot (jump) relations

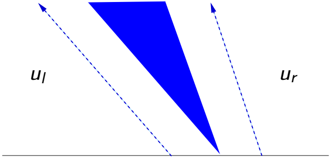
$$\left[\, f(u) \,\right] = s \left[\, u \,\right],$$

- Lax's inequalities:

$$\lambda_i(u_r) < s < \lambda_i(u_l)$$
$$\lambda_{i-1}(u_l) < s < \lambda_{i+1}(u_r).$$

Propagates at speed $s \sim \fint_{u_l}^{u_r} \lambda_i$

Rarefaction waves



Regular solutions,
obtained with integral curves of r_i:

$$\begin{cases} \dfrac{d}{d\sigma} R_i(\sigma) = r_i(R_i(\sigma)), \\ R_i(0) = u_l, \end{cases}$$

with $\sigma \geq 0$.

Propagates at speed $\lambda_i(R_i(\sigma))$

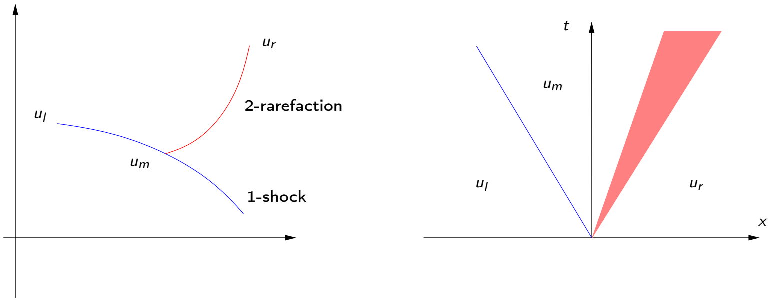
Lax's curves (GNL fields)

- We call $\Phi_i(\cdot, u_l)$ the *i*-th Lax curve consisting of points u_r that can be connected
  - by a *i*-shock
  - or by a *i*-rarefaction wave.
- When $u_+ = \Phi_i(\sigma_i, u_-)$, we call σ_i the strength of the simple wave (u_-, u_+).
- By convention, $\sigma_i > 0$ for rarefactions and $\sigma_i < 0$ for shocks.
- Lax's theorem asserts that for u_l and u_r sufficiently close, one can find (σ_i) such that

$$u_r = \Phi_2(\sigma_2, \cdot) \circ \Phi_1(\sigma_1, \cdot) u_l.$$

- This allows to solve the Riemann problem.

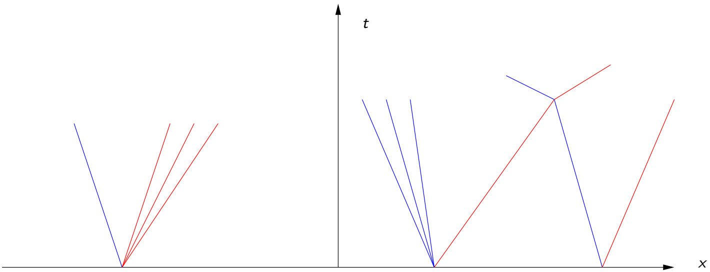
Solving the Riemann problem



- Lax's Theorem proves that one can solve (at least locally) the Riemann problem by first following the 1-curve, then the 2-curve.

Front-tracking algorithm

- Approximate initial condition by piecewise constant functions.
- Solve the Riemann problems and replace rarefaction waves by rarefaction fans.
- For small times, one obtains a piecewise constant function where states are separated by straight lines called fronts.



- At each interaction point (points where fronts meet), iterate the process without splitting again rarefaction fronts

Estimates, convergence, etc.

- One shows than this defines a piecewise constant function, with a finite number of fronts and discrete interaction points.

- A central argument is due to Glimm: consider

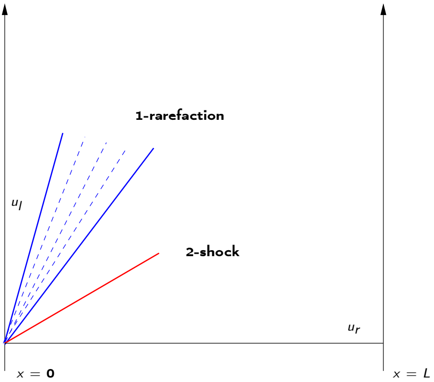
$$V(\tau) = \sum_{\alpha \text{ wave at time } t} |\sigma_\alpha| \; ; \quad Q(\tau) = \sum_{\substack{\alpha,\beta \\ \text{approaching waves}}} |\sigma_\alpha|.|\sigma_\beta|,$$

- Analyzing interactions $\alpha + \beta \to \alpha' + \beta' + \gamma'$ one shows that: for some $C > 0$, if $TV(u_0)$ is small enough, then $V(t) + CQ(t)$ is non-increasing. (Glimm's functional)

- One deduces bounds in $L^\infty_t BV_x$, then in $\text{Lip}_t L^1_x$, so we have compactness...

Boundary Riemann problem

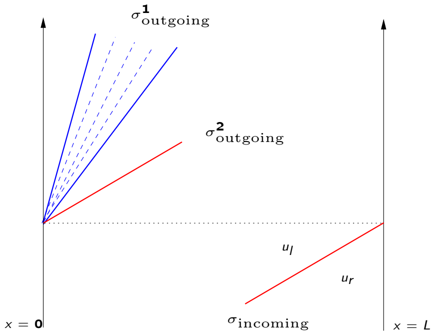
- In our case we have to take the boundary into account, and to be able to solve the boundary Riemann problem.
- Cf. Dubois-LeFloch, Amadori, Amadori-Colombo, Colombo-Guerra, Donadello-Marson, etc.



Boundary "interactions"

- One can then take "boundary interactions" into account.
- One can measure the size of the oungoing fronts in terms of the size of the incoming one. This highly depends on K!
- Roughly speaking, our condition ensures

$$|\sigma^1_{\text{outgoing}}| + |\sigma^2_{\text{outgoing}}| \leq \kappa |\sigma_{\text{incoming}}|, \quad 0 < \kappa < 1.$$



2. Lyapunov functions: the classical case

An example from Coron-Bastin-d'Andrea-Novel: for the system

$$\begin{cases} \partial_t a + c(a,b)\partial_x a = 0, \\ \partial_t b - d(a,b)\partial_x b = 0, \end{cases}$$

with $c, d > 0$, one finds a Lyapunov functional of the form

$$\begin{aligned} \mathcal{L} = & \int_0^L a^2(x) e^{-\mu x}\,dx + \int_0^L b^2(x) e^{+\mu x}\,dx \\ & + C_1 \left(\int_0^L (\partial_x a)^2(x) e^{-\mu x}\,dx + \int_0^L (\partial_x b)^2(x) e^{+\mu x}\,dx \right) \\ & + C_2 \left(\int_0^L (\partial_{xx}^2 a)^2(x) e^{-\mu x}\,dx + \int_0^L (\partial_{xx} b)^2(x) e^{+\mu x}\,dx \right). \end{aligned}$$

This is connected to J.-M. Coron's Lyapunov function for stabilization of the incompressible Euler equation in 2-D simply connected domains (see also G. in the multi-connected case).

3. Our Lyapunov functional

Our Lyapunov functional is as follows:

$$J := V + CQ$$

where

$$\begin{aligned} V(U) &= \sum_{i=0}^{n} \left(|\sigma_{i,1}| + |\sigma_{i,2}|\right) e^{-\gamma x_i}, \\ Q(U) &= \sum_{(x_i,\sigma_i)} |\sigma_i| e^{-\gamma x_i} \left(\sum_{(x_j,\sigma_j) \text{ approaching } (x_i,\sigma_i)} |\sigma_j| e^{-\gamma x_j} \right), \end{aligned}$$

for suitable constants, where

- $\sigma_{i,k}$ is the strength of the k-wave at x_i (σ_i when there is no ambiguity, i.e. for $i \geq 1$),
- $x_1, \ldots, x_n$ are the discontinuities in $(0, L)$,
- $u(t, 0+) = \Psi_2(\sigma_{0,2}, \Psi_1(\sigma_{0,1}, Ku(t, L-)))$.

Our Lyapunov functional, 2

Analyzing in particular interactions of fronts with the boundary, one shows that for suitable constants and provided that

$$TV(u_0) \text{ is small enough,}$$

one has for proper $\nu > 0$:

$$J(t) \leq J(0)\exp(-\nu t).$$

This allows to construct approximations and the solutions globally in time and to get the result.

Open problems

- Considering a less particular case: speeds with different signs, $n \times n$ systems, nonlinear boundary conditions, non GNL characteristic fields, etc.
- Equivalent on networks. We believe this result to be sufficiently similar to the classical case to yield close results on networks in the entropy case.
- What about source terms?