Stability of Non-conservative Systems
Valenciennes, July 4, 2016

A Liapunov function approach to stabilization of coupled hyperbolic systems

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Introduction

In 2002, Fatiha Alabau, Piermarco Cannarsa and Vilmos Komornik published a paper (ACK) entitled

Indirect internal stabilization of weakly coupled evolution equations

in which they investigated the extent of asymptotic stability of the null solution for weakly coupled partially damped equations of the type

$$u'' + A_1 u + Bu' + Cv = v'' + A_2 v + Cu = 0$$

where A_1, A_2, B and C are positive self-adjoint operators satisfying suitable additional conditions.

The main point is that the damping operator acts only on the first component u and when A_1, A_2 are comparable coercive unbounded operators while B, C are coercive and bounded, the coupling is not strong enough to produce an exponential decay in the energy space associated to the conservative part of the system. As a consequence, for initial data in the energy space, the rate of decay is not exponential and explicit decay rates are only available in a weaker norm. Moreover, due to the nature of the result it seems impossible to obtain the asymptotic stability result by the classical Liapunov method consisting in the exhibition of an exponentially decreasing quadratic function of (u, v, u', v').

I. Liapunov's classical approach.

A long time ago (1892), Liapunov defined and investigated the dynamical stability of equilibrium solutions to differential systems of the form

$$U'(t) = F(U(t))$$

where $F \in C^2(\mathbb{R}^N)$. Given $a \in F^{-1}(0)$ he proved that a is asymptotically stable (in fact exponentially stable) as soon as all the eigenvalues of the square matrix $M = DF(a)$ have *negative* real parts. The original proof of Liapunov consisted in considering first the linearized equation

$$Y' = MY(t) \qquad (LIN)$$

for which 0 is an exponentially stable equilibrium.

Under the hypothesis on the eigenvalues, it is not difficult to see that all solutions of (LIN) tend to 0 as t tends to infinity. Then by considering a basis of $\mathbb{R}^N$, it follows easily that for some $T > 0$ we have $||\exp(TM)|| < 1$. Then by a classical division argument we find

$$\forall t \geq 0, \quad ||\exp(tM)|| \leq Ce^{-\delta t}$$

for some $C \geq 1$ and $\delta > 0$, thereby proving exponential stability of 0 for the linearized equation. Is seems that at the time of Liapunov (and even much later) it was not natural to use the potential well argument for the nonlinear perturbation equation by using Duhamel's variation of constants formula. Therefore Liapunov looked for a renorming allowing to get the same estimate with $C = 1$, in which case a direct potential well argument in differential form becomes possible.

The following quadratic function

$$\Phi(z) = \int_0^{\infty} |\exp(sM)z|^2 ds$$

provides a solution of the problem. Indeed for any solution $Y(t) = \exp(tM)Y_0$ of (LIN) we have

$$\frac{d}{dt}\Phi(Y(t)) = \frac{d}{dt}\int_0^{\infty} |\exp(sM)\exp(tM)Y_0|^2 ds$$

$$= \frac{d}{dt}\int_0^{\infty} |\exp(s+t)M)Y_0|^2 ds = \frac{d}{dt}\int_t^{\infty} |\exp(\tau)M)Y_0|^2 d\tau = -|Y(t)|^2$$

By the equivalence of norms on the finite dimensional space $\mathbb{R}^N$ we see immediately that the new norm defined by $||z|| = \Phi(z)^{1/2}$ is a solution.

II. A simple scalar system

A few years ago I was writing down one chapter of [3] devoted to the invariance principle and while looking for simple striking examples I happened to read the (ACK) paper. I realized that the restriction of their equation to the space generated by an eigenfunction already provides an interesting one. The simplest renormalized system can be written in the form :

$$u'' + \lambda u + bu' + cv = v'' + \lambda v + cu = 0 \tag{1}$$

Here $\lambda > 0$, $b > 0$ and $c \neq 0$ can have any sign. In order to have asymptotic stability of $(0, 0, 0, 0)$ in the phase space $\mathbb{R}^4$ we need that the stationary system $\lambda u + cv = \lambda v + cu = 0$ has no solution, hence $c^2 \neq \lambda^2$.

Moreover, standard manipulation gives the identity

$$\frac{d}{dt}(\lambda u^2 + \lambda v^2 + 2cuv + u'^2 + v'^2) = -2bu'^2 \leq 0$$

Assuming $c^2 < \lambda^2$, the function

$$F(u, v, w, z) = \lambda(u^2 + v^2) + 2cuv + w^2 + z^2$$

is a positive definite quadratic form. Since $F(u, v, u', v')$ is non-increasing along the trajectories, the 4 components (u, v, u', v') are bounded and we are in a good position to apply the invariance principle.

Indeed let (u, v) be a solution for which $F(u, v, u', v')$ is constant. Then $2bu'^2 = 0$ implies $u' = 0$, hence u is constant and $u'' = 0$. Then by the first equation $v = -\frac{\lambda}{c} u$ is also constant. Since the only equilibrium is $u = v = 0$ we are done. Now an interesting question occurs : actually we know that the semi-group associated with this ODE system is exponentially stabilizing. In particular the quadratic form Φ introduced by Liapunov is a strict Liapunov function. This form cannot be computed since we do not have access to an explicit formula for the semi-group (the characteristic equation has degree 4 !) We know, however, that the form can be computed on a basis of $\frac{4\times(4+1)}{2} = 10$ monomials in (u, v, w, z). Who is that form ?

III. A Liapunov function for the ODE case

Let us introduce for each solution (u, v) of (1), its total energy

$$\mathcal{E}(u, u', v, v') = \frac{1}{2}\left[u'^2 + v'^2 + \lambda(u^2 + v^2)\right] + cuv$$

Proposition

[H-Jendoubi 2013] Assuming for simplicity $b = 1$, for any $p > 1$ and for all $\varepsilon > 0$ small enough the quadratic form

$$H_\varepsilon = \mathcal{E} - \varepsilon vv' + p\varepsilon uu' + \frac{(p+1)\lambda\varepsilon}{2c}(u'v - uv') \qquad (2)$$

is a strict Liapunov function for (1). More precisely for some $\rho(\varepsilon) > 0$ independent of the solution (u, v) of (1) we have

$$H'_\varepsilon \leq -\rho(\varepsilon)H_\varepsilon \qquad (3)$$

Remark

The only missing term in this quadratic form is $u'v'$. This was predictable since its derivative does not seem to contain any interesting term. Moreover it is usual that the Liapunov function is a small perturbation of the energy. The term in uu' seems to be mandatory since it is what we need in the uncoupled case to produce the emergence of a $-u^2$ term . The term in $-vv'$ is added to produce a $-v'^2$ by differentiation. It is then remarkable that a multiple of the wronskian-like skew product $u'v - uv'$ is sufficient to produce the emergence of a $-v^2$ term and at the same time compensate the “junk terms” coming from the other differentiated terms.

Sketch of proof.

First of all we note that the derivative of the skew product involves $-v^2$. Indeed

$$\frac{d}{dt}(u'v - uv') = (u''v - uv'') = -v(u' + cv + \lambda u) + u(cu + \lambda v)$$

$$= c(u^2 - v^2) - u'v$$

Then we find easily

$$\frac{d}{dt}[-vv' + puu' + \frac{(p+1)\lambda}{2c}(u'v - uv')]$$

$$= pu'^2 - v'^2 + v(cu + \lambda v) - pu(u' + cv + \lambda u)$$

$$+ \frac{(p+1)\lambda}{2}(u^2 - v^2) - \frac{(p+1)\lambda}{2c}u'v$$

$$= pu'^2 - v'^2 - u'(pu + \frac{(p+1)\lambda}{2c}v) - \frac{(p-1)}{2}[\lambda(u^2 + v^2) + 2cuv]$$

The end of the proof is now nearly obvious. First we have

$$\lambda(u^2+v^2)+2cuv \geq (\lambda-|c|)(u^2+v^2)$$

Moreover we have for some constant $K>0$

$$|u'(pu+\frac{(p+1)\lambda}{2c}v)| \leq \frac{(p-1)}{4}(\lambda-|c|)(u^2+v^2)+Ku'^2$$

so that

$$\frac{d}{dt}[-vv'+puu'+\frac{(p+1)\lambda}{2c}(u'v-uv')] \leq (p+K)u'^2-v'^2$$

$$-\frac{(p-1)}{4}(\lambda-|c|)(u^2+v^2)$$

The conclusion follows immediately.

IV. some remarks on the ODE case.

Remark

The ODE case may look a bit simple, however finding this Liapunov function took us some time, and as soon as it was done, infinite dimensional analogs appeared natural. Moreover the ODE case becomes less trivial when we try to use the Liapunov method to get quantitative information on the decay of solutions. For the moment we only have partial results in some very specific range of parameters.

Remark

Although the general solution cannot be computed, it is possible to investigate the rate of decay by another method relying on the characteristic polynomial. For instance for $b = 1$, and for any value of c, it is rather easy to show that the logarithmic decrement of the general solution can never exceed the value $\frac{1}{4}$ while for the uncoupled equation in u the maximum decrement is $\frac{1}{2}$. Indeed the characteristic polynomial P for $b = 1$ is

$$P(\zeta) = (\zeta^2 + \lambda)(\zeta^2 + \zeta + \lambda) - c^2$$

Denoting by ζ_j the 4 characteristic numbers (all positive and eventually counted with their multiplicity) and setting $\rho_j := -Re(\zeta_j)$, the identity $\sum_{j=1}^{4} \rho_j = 1$ implies $\inf_j \rho_j \leq \frac{1}{4}$.

Remark

A sharper study shows that the value $\frac{1}{4}$ is never achieved, but any smaller positive number can be realized for adequate values of $c > 0$ and $\lambda > c$. Recently, we even recovered asymptotically this maximal decrement in a very narrow region by a thorough refinement of our explicit Liapunov function. More precisely we have, assuming for definiteness $c > 0$

Proposition

As $\frac{c}{\lambda}$ tends to 0 and $\frac{c}{\lambda^{1/2}}$ tend to infinity, the logarithmic decrement (as evaluated by the method of proof of Proposition 0.1) tends to the highest possible value $\frac{1}{4}$.

V. The strongly coupled case.

In this section we generalize the scalar system in a framework which concerns finite-dimensional and infinite-dimensional systems as well. Let A be a closed, self-adjoint, positive coercive operator on a separable Hilbert space H. with domain $D(A)$. We denote by (u, v) the inner product of two vectors u, v in H and by $|u|$ the H norm of u. Let $V = D(A^{\frac{1}{2}})$ endowed with the norm given by

$$\forall u \in V, \quad \|u\| = |A^{\frac{1}{2}} u|.$$

The topological dual of H is identified with H, therefore

$$V \subset H = H' \subset V'$$

with continuous and dense imbeddings.

Let $C \in L(V, V')$ satisfy the following conditions

$$\|C\|_{L(V,V')} < 1 \tag{4}$$

$$\ker C = 0, \quad H \subset C(V) \text{ and } C^{-1} \in L(H, V) \tag{5}$$

$$V' \subset C(H) \text{ and } C^{-1} \in L(V', H) \tag{6}$$

$$AC^{-1} - C^{-1}A \in L(H, H) \tag{7}$$

i.e. the operator $AC^{-1} - C^{-1}A \in L(V, V')$ is in fact bounded for the H-norm with values in H and can therefore be extended on the whole of H as a bounded operator. We consider

$$\begin{cases} u'' + u' + Au + Cv = 0, \\ v'' + Av + C^*u = 0. \end{cases} \tag{8}$$

Let us set

$$H_0(u, v, u', v') = \frac{1}{2}(\|u\|^2 + \|v\|^2 + |u'|^2 + |v'|^2) + \langle Cv, u\rangle.$$

Then all solutions of the system (8) are bounded with

$$\frac{d}{dt}H_0(u, v, u', v') = -|u'|^2$$

and we have

Theorem (H-Jendoubi 2013)

For any $p > 1$ and for all $\varepsilon > 0$ small enough the quadratic form $H_\varepsilon = H_\varepsilon(u, v, w, z)$ defined by

$$H_\varepsilon = H_0 - \varepsilon(v, z) + p\varepsilon(u, w) + \frac{(p+1)\varepsilon}{2}[\langle AC^{-1}w, v\rangle - \langle AC^{-1}u, z\rangle]$$

satisfies (3). In particular the semi-group generated by (8) is exponentially damped in $V \times V \times H \times H$.

Example

1. Maximal coupling.

Let Ω be a bounded open domain of $\mathbb{R}^N$. Then for any $\gamma \in (0, 1)$, the system

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u - \gamma \Delta v = 0 \\ \partial_t^2 v - \Delta v - \gamma \Delta u = 0 \end{cases} \tag{9}$$

with homogeneous Dirichlet boundary conditions generates an exponentially damped linear semi-group in $V \times V \times H \times H$ with $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$.

Example

2. Structural coupling.
Ω be a bounded open domain of $\mathbb{R}^N$ with C^2 boundary . Then for any $\gamma \in (0, \lambda_1(\Omega))$, the system

$$\begin{cases} \partial_t^2 u + \Delta^2 u + \partial_t u - \gamma \Delta v = 0 \\ \partial_t^2 v + \Delta^2 v - \gamma \Delta u = 0 \end{cases} \tag{10}$$

with the boundary conditions $u = v = \Delta u = \Delta v = 0$ generates an exponentially damped linear semi-group in $W \times W \times H \times H$ with $H = L^2(\Omega)$ and $W = H^2 \cap H_0^1(\Omega)$.

Example

3. A plate equation with structural non-commuting coupling.

Let Ω be a bounded open domain of $\mathbb{R}^N$ with C^2 boundary and let $m \in L^\infty(\Omega)$ be a non-negative function . Then for any $\gamma \in (0, \lambda_1(\Omega))$, the system

$$\begin{cases} \partial_t^2 u + \Delta^2 u + m(x)u + \partial_t u - \gamma \Delta v = 0 \\ \partial_t^2 v + \Delta^2 v + m(x)v - \gamma \Delta u = 0 \end{cases} \tag{11}$$

with the boundary conditions $u = v = \Delta u = \Delta v = 0$ generates an exponentially damped linear semi-group in $W \times W \times H \times H$ with $H = L^2(\Omega)$ and $W = H^2 \cap H_0^1(\Omega)$.

Assuming that W is endowed with the norm given by

$$\|u\|_W^2 = \int_\Omega (|\Delta u|^2 + m(x)u^2)dx$$

it is easy to check that

$$\|\Delta u\|_{W'} \leq \frac{1}{\lambda_1(\Omega)}\|\Delta u\|_H \leq \frac{1}{\lambda_1(\Omega)}\|u\|_W$$

Moreover, here $C^{-1} = (-\gamma\Delta)^{-1}$ and $A = \Delta^2 + m(x)I$ do not commute, but

$$C^{-1}A - AC^{-1} = C^{-1}\mathcal{M} - \mathcal{M}C^{-1}$$

where $\mathcal{M}$ denotes the operator of multiplication by $m(x)$ is not only bounded, but even compact as an operator from H to itself.

VI. The infinite dimensional weakly coupled case.

We now consider the system

$$\begin{cases} u'' + u' + Au + cv = 0, \\ v'' + Av + cu = 0. \end{cases} \tag{12}$$

and we introduce

$$E(u, v, , w, z) = \frac{1}{2}(\|u\|^2 + \|v\|^2 + |w|^2 + |z|^2) + c(u, v).$$

Then all (weak) solutions of the system (12) are bounded with

$$\frac{d}{dt}E(u, v, u', v') = -|u'|^2$$

and we have

Theorem (H-Jendoubi 2013)

Assume $c \neq 0$ *and* $|c| < \lambda_1(A) := \lambda_1$. *Then for any* $p > \frac{\lambda_1 + |c|}{\lambda_1 - |c|}$ *and all* $\varepsilon > 0$ *small enough the quadratic form* H_ε *defined by*

$$H_\varepsilon(u, v, w, z) = E - \varepsilon\lambda_1(v, z)_* + p\varepsilon(u, w) + \rho\varepsilon[(w, v) - (u, z)]$$

with $\rho = \frac{(p+1)\lambda_1}{2c}$ *satisfies the inequality*

$$\frac{d}{dt} H_\varepsilon(u, v, u', v') \leq -\gamma(p, \varepsilon)\frac{1}{2}(|u|^2 + |v|^2 + \|u'\|_*^2 + \|v'\|_*^2)$$

valid for any weak solution of (12).

Corollary

For any solution (u, v) *of* (12) *we have for some constant* $C > 0$

$$\forall t > 0, \quad |u(t)|^2 + |v(t)|^2 + \|u'(t)\|_*^2 + \|v'(t)\|_*^2 \leq C\frac{E_0}{t}$$

with

$$E_0 = \|u(0)\|^2 + \|v(0)\|^2 + |u'(0)|^2 + |v'(0)|^2.$$

Sketch of proof.

The quadratic form

$$E_{-1}(u,v,,w,z)=\frac{1}{2}(|u|^2+|v|^2+\|u'\|_*^2+\|v'\|_*^2)+c\langle u,v\rangle_*$$

is equivalent to

$$K(u,v,,w,z)=|u|^2+|v|^2+\|u'\|_*^2+\|v'\|_*^2$$

and non-increasing along trajectories. Then we also have

$$\frac{d}{dt}H_\varepsilon(u,v,u',v')\leq -\gamma' E_{-1}(u,v,u',v')$$

and by integrating on $(0,t)$we find

$$tE_{-1}(u,v,u',v')(t)\leq \int_0^t E_{-1}(u,v,u',v')(s)ds\leq CE_0.$$

The result follows immediately.

Remark

We recover here one of the main results of (ACK) by a Liapunov function approach. It seems that many indirect stabilization results can be proved by the same method. All the results involving different usual norms on both sides of the inequality can be deduced from the corollary by using $A-$ invariance, induction or interpolation. The theory will be complete as soon as optimality of the negative power of t is established, and the comparison with similar simpler problems make it look reasonable.

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