Null Controllability for Parabolic Systems with Dynamic Boundary Conditions

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Consider the Parabolic Equation

$$(PE)\quad \begin{cases} y_t - \Delta y + a(t,x)y = v(t,x)1_\omega \ , \ (t,x) \in (0,T) \times \Omega \\ \text{Boundary Conditions,} \\ y(0) = y_0 \end{cases}$$

v is the control acting on a internal regional ω of Ω.

Null controllability of (PE) in time $T > 0$: $\quad \forall y_0 \in L^2(\Omega)$

$$\exists v \in L^2((0,T) \times \omega) \ : \ y(T,\cdot) = 0 \quad \text{on } \Omega.$$

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$$y(T,\cdot) = S(T,0)y_0 + \mathcal{T}v, \qquad \mathcal{T}v = \int_0^T S(T,\tau)\chi_\omega v(\tau)\, d\tau.$$

Hence, the null controllability : $\forall y_0, \exists v : \; y(T,\cdot) = 0$

$$\begin{aligned} &\iff S(T,0)y_0 = -\mathcal{T}v, \\ &\iff ImS(T,0) \subset Im\mathcal{T} \\ &\iff \exists C : \|S(T,0)^*\varphi_T\| \leq C\|\mathcal{T}^*\varphi_T\|, \quad \varphi_T \in L^2(\Omega). \qquad (1) \end{aligned}$$

$$(1) = \int_0^1 \varphi^2(0,x)dx \leq C\int_0^T \int_\omega \varphi^2(t,x)dxdt,$$ **(Observability Inequality**

where φ is the solution of the backward adjoint problem

$$(APE)\begin{cases} -\varphi_t - \Delta\varphi + a(t,x)\varphi = 0 \\ \text{Boundary Conditions,} \\ \varphi(T) = \varphi_T \end{cases}$$

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Carleman estimate

Consider the equation

$$\begin{cases} \varphi_t - \Delta\varphi + a(t,x)\varphi = f(t,x) \\ +BC \\ \varphi(0) = \varphi_0 \end{cases}$$

Carleman estimate :

$$\int_Q \rho_1 \, \varphi^2 + \int_Q \rho_2 \, \varphi_x^2 \leq \int_Q \rho_3 \, f^2 + \int_0^T \int_\omega \rho_4 \varphi^2$$

The Null controllability of this parabolic equations was studied in the literature in the case of boundary conditions :
- Dirichlet,
- Neumann
- Mixed boundary conditions (Robin or Fourier).

-Lebeau-Robbiano
- Fursikov-Imanuvilov

- Albano, Cannarsa, Zuazua, Yamamoto, Zhang, Guerrero, Fernandez-Cara, Puel, Benabdellah, Dermenjian, Le Rousseau, ...

In this work, we are concerned with the null controllability of the following Parabolic equation with dynamic boundary conditions

$$\begin{cases} \partial_t y - d\Delta y + a(t,x)y = \chi_\omega v(t,x) & \text{in } \Omega_T := (0,T)\times\Omega, \\ \partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d\partial_\nu y + b(t,x)y_\Gamma = 0 & \text{on } \Gamma_T := (0,T)\times\Gamma, \\ y(0,\cdot) = y_0, y_\Gamma(0,\cdot) = y_{0,\Gamma}, \end{cases}$$

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\Gamma = \partial\Omega$, $N \geq 2$, and the control region ω is an *arbitrary* nonempty open subset such that $\overline{\omega} \subset \Omega$.
- $y_\Gamma = y|_\Gamma$.
- The term $\partial_t y_\Gamma - \Delta_\Gamma y_\Gamma$ models the tangential diffusive flux on the boundary which is coupled to the equation on the bulk by the normal derivative $\partial_\nu y = (\nu \cdot \nabla y)|_\Gamma$.

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This type of boundary conditions arises for many known equations of mathematical physics.
They are motivated by :

- problems in diffusion phenomena,
- Reaction-diffusion systems in phase-transition phenomena.
- Special flows in hydrodynamics (the flow of heat for a solid in contact with a fluid),
- Models in climatology, ...

References :
C. Gal, Favini, J. and G. Goldstein, Grasselli, Miranville, Meyeries, Romanelli, Vazquez, Zellik, ...

G. R. Goldstein, Derivation of dynamical boundary conditions, Adv. Differential Equations, 11 (2006), 457–480.

The operator Δ_Γ on Γ is given here by the surface divergence theorem

$$\int_\Gamma \Delta_\Gamma y\, z\, dS = -\int_\Gamma \langle \nabla_\Gamma y, \nabla_\Gamma z \rangle_\Gamma \, dS, \; y \in H^2(\Gamma), \; z \in H^1(\Gamma),$$

where ∇_Γ is the surface gradient.

Proposition

The operator $(\Delta_\Gamma, H^2(\Gamma))$ is self-adjoint and non positive on $L^2(\Gamma)$. Thus it generates an analytic C_0-semigroup on $L^2(\Gamma)$.

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Consider the following inhomogeneous parabolic problem with dynamic boundary conditions

$$\begin{cases} \partial_t y - d\Delta y + a(t,x)y = f(t,x), & \text{in } \Omega_T, \\ \partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d(\partial_\nu y)|_\Gamma + b(t,x)y_\Gamma = g(t,x), & \text{on } \Gamma_T \\ y(0,\cdot) = y_0, y_\Gamma(0,\cdot) = y_{0,\Gamma}, \end{cases} \quad (2)$$

On $\mathbb{L}^2 := L^2(\Omega) \times L^2(\Gamma)$, we consider the linear operator

$$A = \begin{pmatrix} d\Delta & 0 \\ -d\partial_\nu & \delta\Delta_\Gamma \end{pmatrix}, \qquad D(A) = \mathbb{H}^2,$$

where $\mathbb{H}^k := \{(y, y_\Gamma) \in H^k(\Omega) \times H^k(\Gamma) : y|_\Gamma = y_\Gamma\}$ for $k \in \mathbb{N}$, Consider the following inhomogeneous parabolic problem with dynamic boundary conditions

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where $\mathbb{H}^k := \{(y, y_\Gamma) \in H^k(\Omega) \times H^k(\Gamma) \,:\, y|_\Gamma = y_\Gamma\}$ for $k \in \mathbb{N}$, Our wellposedness and regularity results for the underlying evolution equations rely on this fact.

Proposition

The operator A is densely defined, self-adjoint, negative and generates an analytic C_0-semigroup $(e^{tA})_{t\geq 0}$ on $\mathbb{L}^2$. We further have $(\mathbb{L}^2, \mathbb{H}^2)_{1/2,2} = \mathbb{H}^1$.

Proof :
Introduce on $\mathbb{L}^2$ the densely defined, closed, symmetric, positive sesquilinear form

$$\mathfrak{a}[y,z] = \int_\Omega d\, \nabla y \cdot \nabla \overline{z}\, dx + \int_\Gamma \delta\, \langle \nabla_\Gamma y, \nabla_\Gamma \overline{z} \rangle_\Gamma\, dS, \qquad D(\mathfrak{a}) = \mathbb{H}^1.$$

It induces a positive self-adjoint sectorial operator $\tilde{A}$ on $\mathbb{L}^2$ and one can show that $A \subset \tilde{A}$.
For the other inclusion, we need to show that $\lambda - A$ is surjective for some "large" λ.

The following perturbed system

$$\begin{cases} \partial_t y - d\Delta y + a(t,x)y = 0 & \text{in } \Omega_T := (0,T)\times\Omega, \\ \partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d(\partial_\nu y)|_\Gamma + b(t,x)y_\Gamma = 0 & \text{on } \Gamma_T := (0,T)\times\Gamma, \\ y(0,\cdot) = y_0, y_\Gamma(0,\cdot) = y_{0,\Gamma}, \end{cases}$$

has also a solution which is an evolution family $S(t,s)$ on $\mathbb{L}^2$ depending strongly continuously on $0 \le s \le t \le T$ such that

$$S(t,\tau)y_0 = e^{(t-\tau)A}y_0 - \int_\tau^t e^{(t-s)A}(a(s,\cdot),b(s,\cdot))S(s,\tau)y_0\,ds$$

Proposition

Let $f \in L^2(\Omega_T)$, $g \in L^2(\Gamma_T)$ *and* $(y_0,) \in \mathbb{L}^2$.

(a) *There is a unique mild solution* $y \in C([0,T];\mathbb{L}^2)$ *of* (2). *The solution map* $(y_0, f, g) \mapsto y$ *is linear and continuous from* $\mathbb{L}^2 \times L^2(\Omega_T) \times L^2(\Gamma_T)$ *to* $C([0,T];\mathbb{L}^2)$. *Moreover,* y *belongs to* $\mathbb{E}_1(\tau, T) := H^1(\tau, T; \mathbb{L}^2) \cap L^2(\tau, T; D(A))$ *and solves* (2) *strongly on* (τ, T) *with initial* $y(\tau)$, *for all* $\tau \in (0, T)$ *and it is given by*

$$y(t) = S(t,0)y_0 + \int_0^t S(t,s)(f(s), g(s))\, ds, \qquad t \in [0,T],$$

Proposition

(b) *Given* $R > 0$*, there is a constant* $C = C(R) > 0$ *such that for all* a *and* b *with* $\|a\|_\infty, \|b\|_\infty \leq R$ *and all data the mild solution of* y *of* (2) *satisfies*

$$\|y\|_{C([0,T];\mathbb{L}^2)} \leq C(\|y_0\|_{\mathbb{L}^2} + \|f\|_{L^2(\Omega_T)} + \|g\|_{L^2(\Gamma_T)}).$$

(c) *If* $y_0 \in \mathbb{H}^1$*, then the mild solution* y *of* (2) *is the strong one, i.e.,* $y \in \mathbb{E}_1 := H^1(0, T; \mathbb{L}^2) \cap L^2(0, T; D(A))$ *and solves* (2) *strongly on* $(0, T)$ *with initial data* y_0*.*

We study the null controllability of the linear system

$$\begin{aligned} \partial_t y - d\Delta y + a(t,x)y &= v(t,x)1_\omega && \text{in } \Omega_T, \\ \partial_t y - \delta\Delta_\Gamma y + d\partial_\nu y + b(t,x)y &= 0 && \text{on } \Gamma_T, \\ y(0,\cdot) &= y_0 && \text{in } \overline{\Omega}, \end{aligned} \tag{3}$$

Definition

The system (3) is said to be null controllable at time $T > 0$ if for all given $y_0 \in L^2(\Omega)$ and $y_{0,\Gamma} \in L^2(\Gamma)$ we can find a control $v \in L^2((0, T) \times \omega)$ such that the solution satisfies

$$y(T, \cdot) = y_\Gamma(T, \cdot) = 0.$$

1. I.I. Vrabie, the approximate controllability of (3), ($\omega = \Omega$).
2. D. Höomberg, K. Krumbiegel, J. Rehberg, Optimal Control of (3), ($\omega = \Omega$.)
3. G. Nikel and Kumpf, Approximate controllability of dynamic boundary control problems, (one-dimension heat equation)

The solution of the linear system

$$\partial_t y - d\Delta y + a(t,x)y = v(t,x)1_\omega \qquad \text{in } \Omega_T, \qquad (4)$$
$$\partial_t y - \delta\Delta_\Gamma y + d\partial_\nu y + b(t,x)y = 0 \qquad \text{on } \Gamma_T, \qquad (5)$$
$$y(0,\cdot) = y_0 \qquad \text{in } \overline{\Omega}, \qquad (6)$$

can be written as

$$y(T,\cdot) = S(T,0)y_0 + \mathcal{T}v, \qquad \mathcal{T}v = \int_0^T S(T,\tau)(\chi_\omega v(\tau),0)\,d\tau.$$

Hence, the null controllability : $\forall y_0, \exists v : \; y(T,\cdot) = 0$

$$\Longleftrightarrow \exists C : \quad \|S(T,0)^*\varphi_T\|_{\mathbb{L}^2} \le C\|\mathcal{T}^*\varphi_T\|_{\mathbb{L}^2}, \quad \varphi_T \in \mathbb{L}^2. \qquad (7)$$

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Lemma

1. The function $\varphi(t) = S(T,t)^*\varphi_T$ *is the solution of the backward adjoint system*

$$\begin{aligned} -\partial_t\varphi - d\Delta\varphi + a(t,x)\varphi &= 0 && \text{in } \Omega_T, \\ -\partial_t\varphi_\Gamma - \delta\Delta_\Gamma\varphi_\Gamma + d\partial_\nu\varphi + b(t,x)\varphi_\Gamma &= 0 && \text{on } \Gamma_T \\ \varphi(T,\cdot) &= \varphi_T && \text{in } \overline{\Omega}, \end{aligned}$$

2. The adjoint of the operator $\mathcal{T}$ *is given by*

$$\mathcal{T}^*\varphi_T = \chi_\omega\varphi.$$

3. The estimate (7) *can be written as (Observability Ineq.)*

$$\|\varphi(0,\cdot)\|_{L^2}^2 + \|\varphi_\Gamma(0,\cdot)\|_{L^2}^2 \le C\int_0^T\int_\omega |\varphi|^2\,dx\,dt$$

The crucial way to show the above observabity inequality is to show a Carleman estimate for the backward adjoint linear problem

$$\begin{aligned} -\partial_t\varphi - d\Delta\varphi + a(t,x)\varphi &= f(t,x) && \text{in } \Omega_T, \\ -\partial_t\varphi_\Gamma - \delta\Delta_\Gamma\varphi_\Gamma + d\partial_\nu\varphi + b(t,x)\varphi_\Gamma &= g(t,x) && \text{on } \Gamma_T \\ \varphi(T,\cdot) &= \varphi_T && \text{in } \overline{\Omega}, \end{aligned} \tag{8}$$

for given φ_T in $H^1(\Omega)$ or in $L^2(\Omega)$, $f \in L^2(\Omega_T)$ and $g \in L^2(\Gamma_T)$.

Lemma

Given a nonempty open set $\omega \Subset \Omega$*, there is a function* $\eta^0 \in C^2(\overline{\Omega})$ *such that*

$$\eta^0 > 0 \quad \text{in } \Omega, \qquad \eta^0 = 0 \quad \text{on } \Gamma, \qquad |\nabla \eta^0| > 0 \quad \text{in } \overline{\Omega \backslash \omega}.$$

Since $|\nabla \eta^0|^2 = |\nabla_\Gamma \eta^0|^2 + |\partial_\nu \eta^0|^2$ on Γ, the function η^0 in the lemma satisfies

$$\nabla_\Gamma \eta^0 = 0, \qquad |\nabla \eta^0| = |\partial_\nu \eta^0|, \qquad \partial_\nu \eta^0 \leq -c < 0 \qquad \text{on } \Gamma \quad (9)$$

for a constant $c > 0$.

Take $\lambda, m > 1$ and η^0 with respect to ω as in the lemma. We define the weight functions α and ξ by

$$\alpha(x,t) = (t(T-t))^{-1}\left(e^{2\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}\right), \quad x \in \overline{\Omega}$$
$$\xi(x,t) = (t(T-t))^{-1} e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}, \quad x \in \overline{\Omega}.$$

Note that the weights are constant on the boundary Γ so that

$$\nabla_\Gamma \alpha = 0 \qquad \text{and} \qquad \nabla_\Gamma \xi = 0 \qquad \text{on } \Gamma. \tag{10}$$

Theorem

There are constants $C > 0$ and $\lambda_1, s_1 \geq 1$ such that, $\forall \lambda \geq \lambda_1, s \geq s_1$ and every mild solution φ of (8), we have

$$
\begin{aligned}
& s\lambda^2 \int_{\Omega_T} e^{-2s\alpha}\xi|\nabla\varphi|^2\,dx\,dt + s^3\lambda^4 \int_{\Omega_T} e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \\
& + s\lambda \int_{\Gamma_T} e^{-2s\alpha}\xi|\nabla_\Gamma\varphi|^2 + s^3\lambda^3 \int_{\Gamma_T} e^{-2s\alpha}\xi^3|\varphi|^2\,dS\,dt \\
& + s\lambda \int_{\Gamma_T} e^{-2s\alpha}\xi|\partial_\nu\varphi|^2\,dS\,dt \\
\leq \quad & Cs^3\lambda^4 \int_0^T \int_\omega e^{-2s\alpha}\xi^3|\varphi|^2\,dx\,dt \\
& + C \int_{\Omega_T} e^{-2s\alpha}|f|^2\,dx\,dt + C \int_{\Gamma_T} e^{-2s\alpha}|g|^2\,dS\,dt.
\end{aligned}
$$

Proof :

We define

$$\psi := e^{-s\alpha}\varphi$$

and rewrite the adjoint equation as

$$M_1\psi + M_2\psi = \tilde{f} \quad \text{in } \Omega_T, \qquad N_1\psi + N_2\psi = \tilde{g} \quad \text{on } \Gamma_T,$$

with the abbreviations

$$\begin{aligned}
M_1\psi &:= \partial_t\psi - 2s\lambda^2\psi\xi|\nabla\eta^0|^2 - 2s\lambda\xi\nabla\psi\cdot\nabla\eta^0,\\
M_2\psi &:= \Delta\psi + s^2\lambda^2\psi\xi^2|\nabla\eta^0|^2 + s\psi\partial_t\alpha,\\
N_1\psi &:= \partial_t\psi + s\lambda\psi\xi\partial_\nu\eta^0\\
N_2\psi &:= \delta\Delta_\Gamma\psi + s\psi\partial_t\alpha - \partial_\nu\psi,\\
\tilde{f} &:= e^{-s\alpha}f + s\lambda\psi\xi\Delta\eta^0 - s\lambda^2\psi\xi|\nabla\eta^0|^2 + a\psi,\\
\tilde{g} &:= e^{-s\alpha}g + b\psi.
\end{aligned}$$

Proof :

$$\sum_{i=1}^{2}[\|M_i\psi\|^2_{L^2(\Omega_T)} + \|N_i\psi\|^2_{L^2(\Gamma_T)}] + s^3\lambda^4\int_{\Omega_T}\xi^3\psi^2\,dx\,dt + s\lambda^2\int_{\Omega_T}\xi|\nabla\psi|$$

$$+ s^3\lambda^3\int_{\Gamma_T}\xi^3\psi^2\,dS\,dt + s\lambda\int_{\Gamma_T}\xi|\nabla_\Gamma\psi|^2\,dS\,dt + s\lambda\int_{\Gamma_T}\xi(\partial_\nu\psi)^2\,dS\,dt$$

$$\leq C\int_{\Omega_T}e^{-2s\alpha}|f|^2\,dx\,dt + C\int_{\Gamma_T}e^{-2s\alpha}|g|^2\,dS\,dt$$

$$+ Cs^3\lambda^4\int_{(0,T)\times\omega}\xi^3\psi^2\,dx\,dt + Cs\lambda^2\int_{(0,T)\times\omega}\xi|\nabla\psi|^2\,dx\,dt$$

$$+ Cs\lambda^2\int_{\Gamma_T}(\partial_\nu\eta^0)^2\xi\psi\partial_\nu\psi\,dS\,dt + Cs\lambda\int_{\Gamma_T}\xi\partial_\nu\eta^0\partial_\nu\psi\psi\,dS\,dt$$

$$+ Cs\lambda\int_{\Gamma_T}\xi\psi|\nabla_\Gamma\partial_\nu\eta^0|\,|\nabla_\Gamma\psi|\,dS\,dt + Cs\lambda\int_{\Gamma_T}\xi|\partial_\nu\eta^0|\,|\nabla_\Gamma\psi|^2\,dS\,dt.$$

For the last integral, we have

$$
\begin{aligned}
Cs\lambda \int_{\Gamma_T} |\partial_\nu \eta^0 \xi| \nabla_\Gamma \psi|^2 \, dS\, dt &\le Cs\lambda \int_0^T \xi \|\nabla_\Gamma \psi\|^2_{L^2(\Gamma)} \, dt \\
&\le C \int_0^T \left(s^{-1/2}\xi^{-1/2}\|\psi\|_{H^2(\Gamma)}\right)\left(s^{3/2}\lambda\xi^{3/2}\|\psi\|_{L^2(\Gamma)}\right) dt \\
&\le \varepsilon s^{-1} \int_{\Gamma_T} \xi^{-1}|\Delta_\Gamma \psi|^2 \, dS\, dt + C_\epsilon s^3\lambda^2 \int_{\Gamma_T} \xi^3 |\psi|^2 \, dS\, dt.
\end{aligned}
$$

We used the interpolation inequality

$$
\|\nabla_\Gamma \psi\|^2_{L^2(\Gamma)} \le C\|\psi\|_{H^2(\Gamma)}\|\psi\|_{L^2(\Gamma)}, \quad \|\cdot\|_{L^2(\Gamma)} + \|\Delta_\Gamma \cdot\|_{L^2(\Gamma)} \equiv \|\cdot\|_{H^2(\Gamma)}.
$$

$$
\delta\Delta_\Gamma \psi = N_2\psi - s\psi\partial_t\alpha + \partial_\nu \psi
$$

.

For the last integral, we have

$$
\begin{aligned}
Cs\lambda \int_{\Gamma_T} |\partial_\nu \eta^0 \xi |\nabla_\Gamma \psi|^2 \, dS\, dt &\le Cs\lambda \int_0^T \xi \|\nabla_\Gamma \psi\|^2_{L^2(\Gamma)} \, dt \\
&\le C \int_0^T \left(s^{-1/2}\xi^{-1/2} \|\psi\|_{H^2(\Gamma)}\right) \left(s^{3/2}\lambda \xi^{3/2} \|\psi\|_{L^2(\Gamma)}\right) dt \\
&\le \varepsilon s^{-1} \int_{\Gamma_T} \xi^{-1} |\Delta_\Gamma \psi|^2 \, dS\, dt + C_\epsilon s^3 \lambda^2 \int_{\Gamma_T} \xi^3 |\psi|^2 \, dS\, dt.
\end{aligned}
$$

We used the interpolation inequality

$$
\|\nabla_\Gamma \psi\|^2_{L^2(\Gamma)} \le C \|\psi\|_{H^2(\Gamma)} \|\psi\|_{L^2(\Gamma)}, \quad \|\cdot\|_{L^2(\Gamma)} + \|\Delta_\Gamma \cdot\|_{L^2(\Gamma)} \equiv \|\cdot\|_{H^2(\Gamma)}.
$$

$$
\delta \Delta_\Gamma \psi = N_2 \psi - s\psi \partial_t \alpha + \partial_\nu \psi
$$

.

Lemma

For $f = g = 0$, we obtain the following fundamental estimates

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\Omega} |\varphi|^2 \, dx \, dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\Gamma} |\varphi_\Gamma|^2 \, dS \, dt \leq C \int_0^T \int_\omega |\varphi|^2 \, dx \, dt$$

and

$$\|\varphi(0,\cdot)\|_{\mathbb{L}^2}^2 \leq C \|\varphi(t,\cdot)\|_{\mathbb{L}^2}^2, \quad 0 \leq t \leq T.$$

Proposition

Let $T > 0$, a nonempty open set $\omega \Subset \Omega$ and $a \in L^\infty(\Omega_T)$ and $b \in L^\infty(\Gamma_T)$. Then there is a constant $C > 0$ (depending on $\Omega, \omega, \|a\|_\infty, \|b\|_\infty$) such that

$$\|\varphi(0,\cdot)\|^2_{L^2(\Omega)} + \|\varphi_\Gamma(0,\cdot)\|^2_{L^2(\Gamma)} \leq C \int_0^T \int_\omega |\varphi|^2 \, dx \, dt$$

for every mild solution φ of the homogeneous backward problem

$$\begin{aligned} -\partial_t\varphi - d\Delta\varphi + a(t,x)\varphi &= 0 && \text{in } \Omega_T, \\ -\partial_t\varphi_\Gamma - \delta\Delta_\Gamma\varphi_\Gamma + d\partial_\nu\varphi + b(t,x)\varphi_\Gamma &= 0 && \text{on } \Gamma_T \\ \varphi(T,\cdot) &= \varphi_T && \text{in } \overline{\Omega}, \end{aligned}$$

Theorem

Let $T > 0$ and coefficients $d, \delta > 0$, $a \in L^\infty(\Omega_T)$ and $b \in L^\infty(\Gamma_T)$ be given. Then for each nonempty open set $\omega \Subset \Omega$ and for all data $y_0, y_{0,\Gamma}$, there is a control $v \in L^2((0,T) \times \omega)$ such that the mild solution y of (4)–(6) satisfies $y(T,\cdot) = y_\Gamma(T,\cdot) = 0$.

Consider the Parabolic equation with dynamic boundary conditions and a control on a part Γ_0 of the boundary Γ

$$\begin{aligned} \partial_t y - d\Delta y + a(t,x)y &= 0, \\ \partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d(\partial_\nu y)|_\Gamma + b(t,x)y_\Gamma &= v1_{\Gamma_0}, \\ y(0,\cdot) = y_0, y_\Gamma(0,\cdot) &= y_{0,\Gamma}. \end{aligned} \tag{11}$$

Proposition

Let $y_0 \in \mathbb{H}^2$ with $y_0 \in W_p^{2-2/p}(\Omega)$ for some $p > (N+2)/2$. Then there is a control $v \in L^2((0,T); L^2_{\mathrm{loc}}(\Gamma_0))$ such that the solution y of (11) satisfies $y(T,\cdot) = 0$ on $\overline{\Omega}$.
This solution is contained in $H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega))$ and has a trace in $H^1(0,T; H^{1/2}(\Gamma')) \cap L^2(0,T; H^{5/2}(\Gamma'))$ where $\Gamma' = (\Gamma \setminus \Gamma_0) \cup \Gamma_1$ for any $\Gamma_1 \Subset \Gamma_0$.

$$\begin{cases} \partial_t y - D\Delta y + Ay = B\chi_\omega v(t,x) & \text{in } (0,T)\times\Omega, \\ \partial_t y_\Gamma - D_\Gamma \Delta_\Gamma y_\Gamma + D(\partial_\nu y)|_\Gamma + A_\Gamma(t,x) y = 0 & \text{in } (0,T)\times\Gamma, \\ (y, y_\Gamma)|_{t=0} = (y_0, y_{0,\Gamma}) & \text{in } \Omega\times\Gamma, \end{cases} \tag{12}$$

where $A = (a_{ij})_{1\le i,j\le n}$, $A_\Gamma = (a^\Gamma_{ij})_{1\le i,j\le n}$ and $D = diag(d, \cdots, d)$, $D_\Gamma = diag(\delta, \cdots, \delta)$, B is a $n\times m$ matrix, $v = (v_1, \cdots, v_m)^*$.

The aim : The system (12) is null controllable iff

$$rank[B, AB, \ldots, A^{n-1}B] = n. \tag{13}$$

This question has been extensively studied, in the case of **Static boundary conditions**, and several optimal results are obtained by : *Ammar-Khodja, Benabdellah, de Teresa, Dupaix, Dermenjian, Fernandez-Cara, González-Burgos, ...*

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Theorem

Let $T > 0$, $\omega \Subset \Omega$ be nonempty and open,
$A \in \mathcal{L}(\mathbb{R}^n), B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ *such that* (13) *holds, and*
$C \in (L^\infty(\Gamma_T))^{n^2}$.
Then, system (12) *is null controllable on* $[0, T]$.

Proposition

Let $T > 0$, $\omega \subset \Omega$ be nonempty and open, $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ such that (15) *holds. There is a constant $C > 0$ such that for all $\varphi_T \in (\mathbb{L}^2)^n$ the mild solution φ of the adjoint system of* (12) *satisfies the* **Observability Inequality**

$$\|\varphi(0, \cdot)\|^2_{(\mathbb{L}^2)^n} \le C \int_{\omega_T} |B^*\varphi|^2 \, dx \, dt. \tag{14}$$

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Theorem

Let $T > 0$, $\omega \Subset \Omega$ be nonempty and open, A and B satisfy the condition (13) and $C \in L^\infty(\Gamma_T)^{n^2}$. Then, there exists $\hat{\lambda} > 0$ and $l \geq 3$ such that for every $\lambda \geq \hat{\lambda}$, we can choose $s_0(\lambda, l)$ satisfying : there is a constant $C(\lambda, l) > 0$ such that every solution φ of of the adjoint system of (12) satisfies

$$\sum_{i=1}^{n} J(3(n-i), \varphi_i) \leq Cs^l \int_{\omega\times(0,T)} \gamma^l e^{-2s\alpha} |B^*\varphi|^2 \, dx \, dt$$

for all $s \geq s_0(\lambda, l)$. The term $J(k, z)$ is given by

$$J(k,z) = s^{k+1} \int_Q \gamma^{k+1} e^{-2s\alpha} |\nabla z|^2 \, dx \, dt + s^{k+1} \int_{\Gamma_T} \gamma^{k+1} e^{-2s\alpha} |\nabla_\Gamma z|^2$$

$$+ s^{k+3} \int_Q \gamma^{k+3} e^{-2s\alpha} |z|^2 \, dx \, dt + s^{k+3} \int_{\Gamma_T} \gamma^{k+3} e^{-2s\alpha} |z|^2 \, dS \, dt$$

– The necessary conditions of the null controllability of the above systems can be obtained in the particular case :

$$A_\Gamma = A.$$

– The General case $A_\Gamma \neq A$ is open.

– The case of $D = diag(d_1, \ldots, d_n)$ and $D_\Gamma = diag(\delta_1, \ldots, \delta_n)$ is also an open problem.

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System (12) (with static boundary case)is null controllable iff

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In this work, the regularity of the surface diffusion $\delta\Delta_\Gamma$ in the boundary equation played a crucial role to absorb some boundary integrals.

In the case $\delta = 0$, i.e.,

$$\partial_t y - D\Delta y + Ay = v1_\omega,$$
$$\partial_t y_\Gamma + D(\partial_\nu y)|_\Gamma + Cy_\Gamma = 0$$
$$y(0,\cdot) = y_0, y_\Gamma(0,\cdot) = y_{0,\Gamma}.$$

we could not show a Carleman estimate ! !.
- Could we show a uniform Carleman estimate on δ and tend δ to 0 ? ?
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Thank you very much for your attention