Stability of difference equations and applications to transport and wave propagation on networks

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joint work with Yacine Chitour and Mario Sigalotti

Stability of non-conservative systems
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Outline

1. Introduction
2. Stability analysis
3. Application to a transport system
4. Relative controllability

Introduction

Linear difference equations

$$\Sigma(\Lambda, A) : \qquad x(t) = \sum_{j=1}^{N} A_j(t) x(t - \Lambda_j), \qquad t \geq 0.$$

- $\Lambda_1, \dots, \Lambda_N$: positive delays.
- $A_1(t), \dots, A_N(t)$: time-dependent $d \times d$ matrices.
- $x(t) \in \mathbb{C}^d$.
- Notation: $\Lambda_{\min} = \min_i \Lambda_i$, $\Lambda_{\max} = \max_i \Lambda_i$.

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Motivation:

- Applications to some hyperbolic PDEs.
- Generalization of previous results: $N = 1$, autonomous.

Introduction

Motivation: transport systems

Hyperbolic PDEs → difference equations: [Cooke, Krumme; 1968], [Slemrod; 1971], [Greenberg, Li; 1984], [Coron, Bastin, d'Andréa Novel; 2008], [Fridman, Mondié, Saldivar; 2010], [Gugat, Sigalotti; 2010]...

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$$\begin{cases} \partial_t u_i(t,\xi) + \partial_\xi u_i(t,\xi) + \alpha_i(t,\xi) u_i(t,\xi) = 0, \\ \qquad\qquad t \in \mathbb{R}_+,\ \xi \in [0, \Lambda_i],\ i \in [\![1, N]\!], \\ u_i(t,0) = \displaystyle\sum_{j=1}^N m_{ij}(t) u_j(t, \Lambda_j), \quad t \in \mathbb{R}_+,\ i \in [\![1, N]\!]. \end{cases}$$

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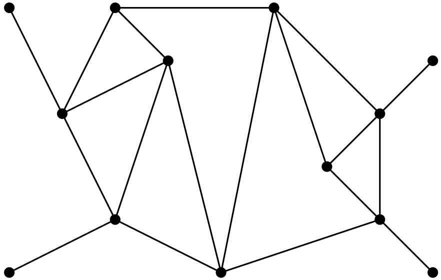
Method of characteristics: for $t \geq \Lambda_{\max}$,

$$u_i(t, 0) = \sum_{j=1}^N m_{ij}(t) u_j(t, \Lambda_j) = \sum_{j=1}^N m_{ij}(t) e^{-\int_0^{\Lambda_j} \alpha_j(t-s, \Lambda_j - s) ds} u_j(t - \Lambda_j, 0).$$

Set $x(t) = (u_i(t, 0))_{i \in [\![1, N]\!]}$. Then x satisfies a difference equation.

Introduction

Motivation: wave propagation on networks



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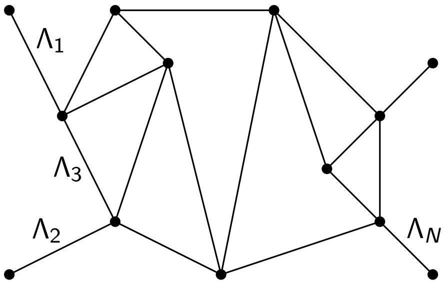

Edges: $\mathcal{E}$
Vertices: $\mathcal{V}$

$$\partial_{tt}^2 u_i(t, \xi) = \partial_{\xi\xi}^2 u_i(t, \xi)$$
$$u_i(t, q) = u_j(t, q), \qquad \forall q \in \mathcal{V}, \ \forall i, j \in \mathcal{E}_q$$

$+$ conditions on vertices.

Introduction

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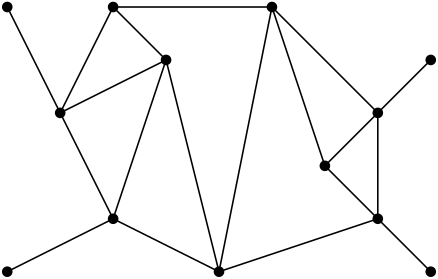

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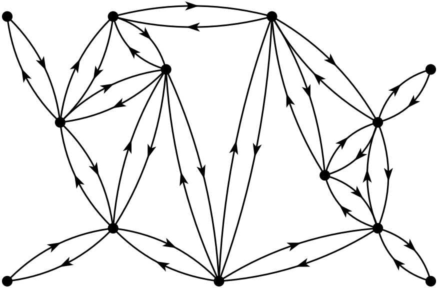
Motivation: wave propagation on networks

D'Alembert decomposition on travelling waves:



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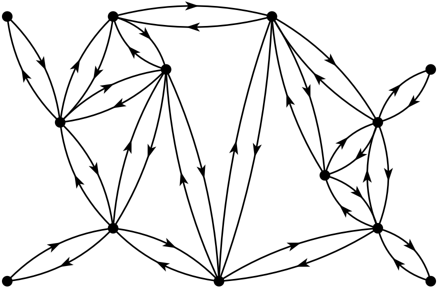

System of $2N$ transport equations.

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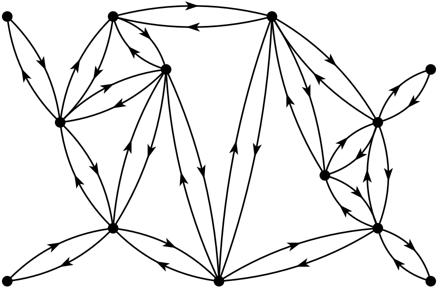

System of $2N$ transport equations.

Can be reduced to a system of difference equations.

Introduction

Motivation: case $N = 1$

- When $N = 1$: $x(t) = A(t)x(t - \Lambda)$.
- Can be reduced to $x_n = A_n x_{n-1}$.

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Autonomous system

$$x_n = Ax_{n-1}$$
$$A \in \mathcal{M}_d(\mathbb{C})$$

Exponential stability $\iff \rho(A) < 1$

Finite-time stability $\iff \rho(A) = 0$

$$\rho(A) = \lim_{n \to +\infty} \left|A^n\right|^{\frac{1}{n}} = \max_{\lambda \in \sigma(A)} \left|\lambda\right|$$

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$$x_n = Ax_{n-1}$$
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Arbitrary switching

$$x_n = A_n x_{n-1}$$
$$A_n \in \mathfrak{B} \subset \mathcal{M}_d(\mathbb{C})$$

Exponential stability $\iff \rho(A) < 1$

Finite-time stability $\iff \rho(A) = 0$

Uniform exponential stability $\iff \rho_{\mathrm{J}}(\mathfrak{B}) < 1$

Finite-time stability $\iff \rho_{\mathrm{J}}(\mathfrak{B}) = 0$

$$\rho(A) = \lim_{n \to +\infty} \left|A^n\right|^{\frac{1}{n}} = \max_{\lambda \in \sigma(A)} |\lambda|$$

$$\rho_{\mathrm{J}}(\mathfrak{B}) = \lim_{n \to +\infty} \sup_{A_1, \dots, A_n \in \mathfrak{B}} \left|A_1 A_2 \cdots A_n\right|^{\frac{1}{n}}$$

(cf. [Jungers; 2009])

Introduction

Motivation: autonomous case

$$\Sigma^{\text{aut}}(\Lambda, A) : \qquad x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j), \qquad t \geq 0$$

- [Cruz, Hale; 1970], [Henry; 1974], [Michiels et al.; 2009]...
- Studied through spectral methods.
- Stability: real parts of the roots of $\det \left(\mathrm{Id} - \sum_{j=1}^{N} A_j e^{-s\Lambda_j}\right) = 0$ (*exponential polynomial*, see [Avellar, Hale; 1980]).

Introduction

Motivation: autonomous case

Let $\rho_{\mathrm{HS}}(A) = \max_{(\theta_1, \dots, \theta_N) \in [0, 2\pi]^N} \rho\left(\sum_{j=1}^N A_j e^{i \theta_j}\right)$.

Theorem ([Hale; 1975], [Silkowski; 1976])

The following are equivalent:

- $\rho_{\mathrm{HS}}(A) < 1$;
- $\Sigma^{\mathrm{aut}}(\Lambda, A)$ *is exponentially stable for some* $\Lambda \in (0, +\infty)^N$ *with rationally independent components;*
- $\Sigma^{\mathrm{aut}}(\Lambda, A)$ *is exponentially stable for every* $\Lambda \in (0, +\infty)^N$.

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- Still true if we replace $\rho_{\mathrm{HS}}(A) < 1$ by $\rho_{\mathrm{HS}}(A) = 0$ and exponential by finite-time stability.
- For rationally dependent delays: [Michiels et al.; 2009].
- Can this be generalized to the non-autonomous case?

Introduction
Main problem

Main problem: exponential stability of the non-autonomous system $\Sigma(\Lambda, A)$ uniformly with respect to a given class $\mathcal{A}$ of functions $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$.

- The techniques from the autonomous case cannot be applied.
- Our approach: explicit formula for solutions of $\Sigma(\Lambda, A)$.
- When $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$, we obtain a generalization of Hale–Silkowski's Theorem.

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Exponential stability criteria:

	Autonomous	Arbitrary switching
$N = 1$	$\rho(A) < 1$	$\rho_J(\mathfrak{B}) < 1$
any N	$\rho_{HS}(A) < 1$	

Stability analysis

Explicit solution (I)

$$\Sigma(\Lambda, A): \qquad x(t) = \sum_{j=1}^{N} A_j(t) x(t - \Lambda_j), \qquad t \geq 0$$

Solution with initial condition $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$: x satisfying $\Sigma(\Lambda, A)$ for $t \geq 0$ and $x(t) = x_0(t)$ for $-\Lambda_{\max} \leq t < 0$.

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Lemma

The solution $x : [-\Lambda_{\max}, +\infty) \to \mathbb{C}^d$ of $\Sigma(\Lambda, A)$ with initial condition $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$ is, for $t \geq 0$,

$$x(t) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ t < \Lambda \cdot \mathbf{n} \leq t + \Lambda_{\max}}} \sum_{\substack{j \in [\![1, N]\!] \\ \Lambda \cdot \mathbf{n} - \Lambda_j \leq t}} \Xi_{\mathbf{n} - e_j, t}^{\Lambda, A} A_j(t - \Lambda \cdot \mathbf{n} + \Lambda_j) x_0(t - \Lambda \cdot \mathbf{n}),$$

where the matrices $\Xi_{\mathbf{n}, t}^{\Lambda, A}$ are defined recursively by

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Stability analysis (I)

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- $\mathsf{X}_p = L^p([-\Lambda_{\max}, 0], \mathbb{C}^d)$, $p \in [1, +\infty]$
- $\mathcal{A}$: set of uniformly locally bounded functions taking values in N-tuples of matrices
- $\Sigma(\Lambda, \mathcal{A})$: family of systems $\Sigma(\Lambda, A)$ for $A \in \mathcal{A}$.
- For x solution of $\Sigma(\Lambda, A)$, $x_t = x(t + \cdot)|_{[-\Lambda_{\max}, 0]} \in \mathsf{X}_p$.

Stability analysis

Stability analysis (I)

Definition

$\Sigma(\Lambda, \mathcal{A})$ is of:

- *exponential type* γ in X_p if $\forall \varepsilon > 0\ \exists K > 0$ s.t. $\forall A \in \mathcal{A}$, $\forall x_0 \in \mathsf{X}_p$, the solution x satisfies $\left\lVert x_t \right\rVert_{\mathsf{X}_p} \leq K e^{(\gamma + \varepsilon) t} \left\lVert x_0 \right\rVert_{\mathsf{X}_p}$;
- *Θ-exponential type* γ if $\forall \varepsilon > 0\ \exists K > 0$ s.t. $\forall A \in \mathcal{A}$, $\forall \mathbf{n} \in \mathbb{N}^N$, a.e. $t \in (\Lambda \cdot \mathbf{n} - \Lambda_{\max}, \Lambda \cdot \mathbf{n})$, one has $\left\lvert \Theta_{\mathbf{n}, t}^{\Lambda, A} \right\rvert \leq K e^{(\gamma + \varepsilon) t}$;
- *Ξ-exponential type* γ if $\forall \varepsilon > 0\ \exists K > 0$ s.t. $\forall A \in \mathcal{A}$, $\forall \mathbf{n} \in \mathbb{N}^N$, a.e. $t \in \mathbb{R}$, one has $\left\lvert \Xi_{\mathbf{n}, t}^{\Lambda, A} \right\rvert \leq K e^{(\gamma + \varepsilon) \Lambda \cdot \mathbf{n}}$.

Exponential stability: exponential type $\gamma < 0$.

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Theorem (Chitour, M., Sigalotti; 2015)

Let $\Lambda \in (0, +\infty)^N$ and $\mathcal{A}$ be uniformly locally bounded.

- *If $\Sigma(\Lambda, \mathcal{A})$ is of Θ-exponential type γ then $\forall p \in [1, +\infty]$ it is of exponential type γ in X_p.*

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- *Suppose that $\Lambda_1, \dots, \Lambda_N$ are rationally independent. If $\exists p \in [1, +\infty]$ such that $\Sigma(\Lambda, \mathcal{A})$ is of exponential type γ in X_p, then it is of Θ-exponential type γ.*

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- *Suppose that $\Lambda_1, \dots, \Lambda_N$ are rationally independent. If $\exists p \in [1, +\infty]$ such that $\Sigma(\Lambda, \mathcal{A})$ is of exponential type γ in X_p, then it is of Θ-exponential type γ.*
- *Suppose that $\mathcal{A}$ is shift-invariant. Then Θ- and Ξ-exponential types γ are equivalent.*

Stability analysis

Rational dependence of the delays

Let $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$. We define

$$Z(\Lambda) = \{\mathbf{n} \in \mathbb{Z}^N \mid \Lambda \cdot \mathbf{n} = 0\},$$

$$V(\Lambda) = \{L \in \mathbb{R}^N \mid Z(\Lambda) \subset Z(L)\}, \qquad \textit{(more rationally dependent)}$$

$$W(\Lambda) = \{L \in \mathbb{R}^N \mid Z(\Lambda) = Z(L)\}, \qquad \textit{(as rationally dependent)}$$

$$V_+(\Lambda) = V(\Lambda) \cap (0, +\infty)^N, \qquad W_+(\Lambda) = W(\Lambda) \cap (0, +\infty)^N.$$

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Example: $\Lambda = (1, \sqrt{2}, 1 + \sqrt{2})$.

- $Z(\Lambda) = \{(n, m, -n - m) \mid n, m \in \mathbb{Z}\}$;
- $V(\Lambda) = \{(a, b, a + b) \mid a, b \in \mathbb{R}\}$;
- $W(\Lambda) = \{(a, b, a + b) \mid a, b \in \mathbb{R}$ rationally independent$\}$.

Stability analysis

Rational dependence of the delays

For $\Lambda \in (0, +\infty)^N$, define the following equivalence relations on $[\![1, N]\!]$ and $\mathbb{Z}^N$,

$$i \sim j \text{ iff } \Lambda_i = \Lambda_j, \qquad \mathbf{n} \approx \mathbf{n}' \text{ iff } \Lambda \cdot \mathbf{n} = \Lambda \cdot \mathbf{n}',$$
$$\mathcal{J} = [\![1, N]\!] / \sim, \qquad \mathcal{Z} = \mathbb{Z}^N / \approx .$$

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For $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, $L \in V_+(\Lambda)$, $[\mathbf{n}] \in \mathcal{Z}$, $[i] \in \mathcal{J}$, and $t \in \mathbb{R}$,

$$\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{\mathbf{n}' \in [\mathbf{n}]} \Xi_{\mathbf{n}',t}^{L,A}, \qquad \widehat{A}_{[i]}^{\Lambda}(t) = \sum_{j \in [i]} A_j(t),$$
$$\widehat{\Theta}_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{\substack{[j] \in \mathcal{J} \\ L \cdot \mathbf{n} - L_j \leq t}} \widehat{\Xi}_{[\mathbf{n} - e_j],t}^{L,\Lambda,A} \widehat{A}_{[j]}^{\Lambda}(t - L \cdot \mathbf{n} + L_j).$$

Stability analysis
Explicit solution (II)

Lemma (Chitour, M., Sigalotti; 2015)

Let $\Lambda \in (0, +\infty)^N$, $L \in V_+(\Lambda)$, $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$, *and* $x_0 : [-L_{\max}, 0) \to \mathbb{C}^d$. *The corresponding solution* $x : [-L_{\max}, +\infty) \to \mathbb{C}^d$ *of* $\Sigma(L, A)$ *is, for* $t \geq 0$,

$$x(t) = \sum_{\substack{[\mathbf{n}] \in \mathcal{Z} \\ t < L \cdot \mathbf{n} \leq t + L_{\max}}} \sum_{\substack{[j] \in \mathcal{J} \\ L \cdot \mathbf{n} - L_j \leq t}} \widehat{\Xi}_{[\mathbf{n} - e_j], t}^{L, \Lambda, A} \widehat{A}_{[j]}^{\Lambda}(t - L \cdot \mathbf{n} + L_j) x_0(t - L \cdot \mathbf{n})$$

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Stability analysis

Stability analysis (II)

We can define exponential types for $\widehat{\Theta}$ and $\widehat{\Xi}$ similarly. Since they depend on Λ, we write $(\widehat{\Theta}, \Lambda)$- and $(\widehat{\Xi}, \Lambda)$-*exponential types*.

Stability analysis
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Theorem (Chitour, M., Sigalotti; 2015)

Let $\Lambda \in (0, +\infty)^N$ and $\mathcal{A}$ be uniformly locally bounded.

- *Let $L \in V_+(\Lambda)$. If $\Sigma(L, \mathcal{A})$ is of $(\widehat{\Theta}, \Lambda)$-exponential type γ then $\forall p \in [1, +\infty]$ it is of exponential type γ in X_p.*

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- *Let $L \in W_+(\Lambda)$. If $\exists p \in [1, +\infty]$ such that $\Sigma(L, \mathcal{A})$ is of exponential type γ in X_p, then it is of $(\widehat{\Theta}, \Lambda)$-exponential type γ.*
- *Suppose that $\mathcal{A}$ is shift-invariant. Then $(\widehat{\Theta}, \Lambda)$- and $(\widehat{\Xi}, \Lambda)$-exponential types γ are equivalent.*

Stability analysis
Maximal Lyapunov exponent

Definition

The *maximal Lyapunov exponent of $\Sigma(L, \mathcal{A})$* in X_p is

$$\lambda_p(L, \mathcal{A}) = \limsup_{t \to +\infty} \sup_{A \in \mathcal{A}} \sup_{\substack{x_0 \in \mathsf{X}_p \\ \|x_0\|_{\mathsf{X}_p} = 1}} \frac{\log \|x_t\|_{\mathsf{X}_p}}{t}.$$

Stability analysis

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Proposition

$\lambda_p(L, \mathcal{A}) = \inf\{\gamma \in \mathbb{R} \mid \Sigma(L, \mathcal{A})$ is of exponential type γ in $\mathsf{X}_p\}$.
In particular,

$$\Sigma(L, \mathcal{A}) \text{ exponentially stable } \iff \lambda_p(L, \mathcal{A}) < 0.$$

By the previous results, $\lambda_p(L, \mathcal{A})$ does not depend on p.

Stability analysis

Maximal Lyapunov exponent

Theorem (Chitour, M., Sigalotti; 2015)

Let $\Lambda \in (0, +\infty)^N$ and suppose that $\mathcal{A}$ is shift-invariant. For every $L \in W_+(\Lambda)$ and $p \in [1, +\infty]$,

$$\lambda_p(L, \mathcal{A}) = \limsup_{\left|\mathbf{n}\right|_1 \to +\infty} \sup_{A \in \mathcal{A}} \operatorname*{ess\,sup}_{t \in \mathbb{R}} \frac{\log \left|\widehat{\Xi}_{\mathbf{n}, t}^{L, \Lambda, A}\right|}{L \cdot \mathbf{n}}.$$

Stability analysis

Arbitrary switching

$$\Sigma(L, A): \qquad x(t) = \sum_{j=1}^{N} A_j(t) x(t - L_j), \qquad t \geq 0.$$

- $\mathfrak{B} \subset \mathcal{M}_d(\mathbb{C})^N$: bounded set of N-tuples of matrices.
- $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$.
- $(A_1(t), \dots, A_N(t))$ is any measurable function taking values on $\mathfrak{B}$: switched system with arbitrary switching signal.
- In this case, one can obtain more precise results.

Stability analysis

Arbitrary switching

Using the recurrence relation for $\Xi_{\mathbf{n},t}^{L,A}$, we obtain:

$$\widehat{\Xi}_{[\mathbf{n}],t}^{L,\Lambda,A} = \sum_{\mathbf{n}' \in [\mathbf{n}] \cap \mathbb{N}^N} \sum_{v \in V_{\mathbf{n}'}} \prod_{k=1}^{|\mathbf{n}'|_1} A_{v_k} \left(t - \sum_{r=1}^{k-1} L_{v_r}\right).$$

$V_{\mathbf{n}}$: set of all permutations of $(\underbrace{1, \dots, 1}_{n_1 \text{ times}}, \underbrace{2, \dots, 2}_{n_2 \text{ times}}, \dots, \underbrace{N, \dots, N}_{n_N \text{ times}})$.

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Definition

$$\mu(\Lambda, \mathfrak{B}) = \limsup_{\substack{\xi \to +\infty \\ \xi \in \mathcal{L}(\Lambda)}} \sup_{\substack{B^r \in \mathfrak{B} \\ \text{for } r \in \mathcal{L}_\xi(\Lambda)}} \left| \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} = \xi}} \sum_{v \in V_{\mathbf{n}}} \prod_{k=1}^{|\mathbf{n}|_1} B_{v_k}^{\Lambda_{v_1} + \dots + \Lambda_{v_{k-1}}} \right|^{\frac{1}{\xi}},$$

where $\mathcal{L}(\Lambda) = \{\Lambda \cdot \mathbf{n} \mid \mathbf{n} \in \mathbb{N}^N\}$ and $\mathcal{L}_\xi(\Lambda) = \mathcal{L}(\Lambda) \cap [0, \xi)$.

Stability analysis

Arbitrary switching

Theorem (Chitour, M., Sigalotti; 2015)

- $\lambda_p(\Lambda, \mathcal{A}) = \log \mu(\Lambda, \mathfrak{B})$;

Stability analysis
Arbitrary switching

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- $\lambda_p(\Lambda, \mathcal{A}) = \log \mu(\Lambda, \mathfrak{B})$;
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- *for every* $L \in W_+(\Lambda)$, $m_2 \lambda_p(\Lambda, \mathcal{A}) \leq \lambda_p(L, \mathcal{A}) \leq m_1 \lambda_p(\Lambda, \mathcal{A})$.

Here, $\{m_1, m_2\} = \left\{\min_{j \in [\![1, N]\!]} \frac{\Lambda_j}{L_j}, \max_{j \in [\![1, N]\!]} \frac{\Lambda_j}{L_j}\right\}$.

Stability analysis
Arbitrary switching

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Corollary

The following statements are equivalent:

- $\mu(\Lambda, \mathfrak{B}) < 1$;
- $\Sigma(\Lambda, \mathcal{A})$ *is exponentially stable in* X_p *for some* $p \in [1, +\infty]$;
- $\Sigma(L, \mathcal{A})$ *is exponentially stable in* X_p *for every* $p \in [1, +\infty]$ *and* $L \in V_+(\Lambda)$.

Stability analysis
Conclusion

Exponential stability criteria:

	Autonomous	Arbitrary switching
$N = 1$	$\rho(A) < 1$	$\rho_{\mathrm{J}}(\mathfrak{B}) < 1$
any N	$\rho_{\mathrm{HS}}(A) < 1$	$\mu(\Lambda, \mathfrak{B}) < 1$

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Interesting questions:

- Both $\rho(A)$ and $\rho_{\mathsf{J}}(\mathfrak{B})$ are limits and $\lim_{n \to +\infty}$ can be replaced by $\inf_{n \in \mathbb{N}^*}$. Is the same true for $\mu(\Lambda, \mathfrak{B})$?

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- $\rho(A) = 0$, $\rho_{\mathrm{J}}(\mathfrak{B}) = 0$, and $\rho_{\mathrm{HS}}(A) = 0$ are equivalent to convergence in finite time. Is this also true for $\mu(\Lambda, \mathfrak{B})$?

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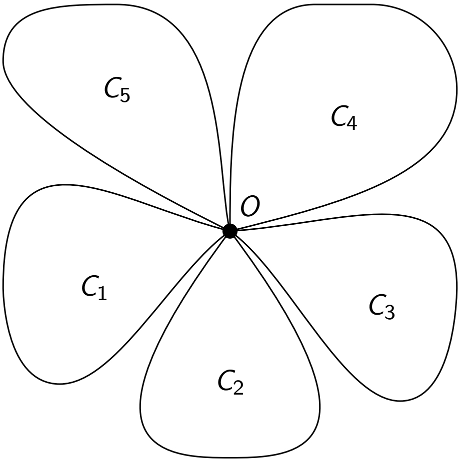
- Can we numerically compute or approximate μ? (For ρ_{J}, this problem is NP-hard, Turing-undecidable, and non-algebraic, but several useful bounds and approximations exist, see [Jungers; 2009]).
- What can we say if $\Lambda_1, \ldots, \Lambda_N$ are time-dependent?

Application to a transport system
Transport system



Application to a transport system

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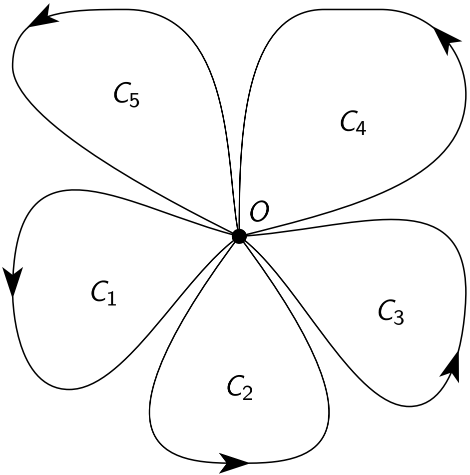

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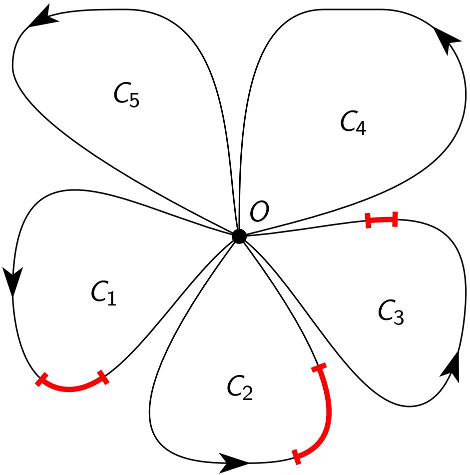

Application to a transport system

Transport system

$$\begin{cases} \partial_t u_i(t,\xi) + \partial_\xi u_i(t,\xi) + \alpha_i(t)\chi_i(\xi)u_i(t,\xi) = 0, \\ \qquad\qquad t \in \mathbb{R}_+,\ \xi \in [0, L_i],\ i \in [\![1, N_d]\!], \\ \partial_t u_i(t,\xi) + \partial_\xi u_i(t,\xi) = 0, \quad t \in \mathbb{R}_+,\ \xi \in [0, L_i],\ i \in [\![N_d + 1, N]\!], \\ u_i(t,0) = \sum_{j=1}^{N} m_{ij} u_j(t, L_j), \quad t \in \mathbb{R}_+,\ i \in [\![1, N]\!]. \end{cases}$$

- χ_i: characteristic function of an interval $[a_i, b_i] \subset [0, L_i]$.
- $M = (m_{ij})_{1 \leq i,j \leq N}$: transmission matrix.
- α_i is *persistently exciting* for $i \in [\![1, N_d]\!]$.

Application to a transport system

Persistence of excitation

- Persistently exciting (PE) signals: for $T \geq \mu > 0$, we say that $\alpha \in \mathcal{G}(T, \mu)$ if $\alpha \in L^\infty(\mathbb{R}; [0, 1])$ and
$$\forall t \in \mathbb{R}, \quad \int_t^{t+T} \alpha(s) ds \geq \mu.$$
- $\mathcal{G}(T, \mu)$ is shift-invariant.

Application to a transport system

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- $\mathcal{G}(T, \mu)$ is shift-invariant.
- Introduced in the context of identification and adaptive control [Anderson; 1977].
- Much studied in finite-dimensional control systems [Chitour, Sigalotti; 2010], [Chitour, M., Sigalotti; 2013].

Application to a transport system
Main result

Hypotheses:

- There exist $i, j \in [\![1, N]\!]$ such that $\frac{L_i}{L_j} \notin \mathbb{Q}$.
- $\left|M\right|_1 \leq 1$ and $m_{ij} \neq 0$ for every $i, j \in [\![1, N]\!]$.

Theorem

$\forall T \geq \mu > 0$, $\exists C, \gamma > 0$ s.t., $\forall p \in [1, +\infty]$, $\forall u_{i,0} \in L^p(0, L_i)$, $i \in [\![1, N]\!]$, and $\forall \alpha_k \in \mathcal{G}(T, \mu)$, $k \in [\![1, N_d]\!]$, the corresponding solution satisfies

$$\sum_{i=1}^N \left\|u_i(t)\right\|_{L^p(0, L_i)} \leq C e^{-\gamma t} \sum_{i=1}^N \left\|u_{i,0}\right\|_{L^p(0, L_i)}, \qquad \forall t \geq 0.$$

Application to a transport system
Technique of the proof

- For $t \geq L_{\max}$:

$$u_i(t, 0) = \sum_{j=1}^N m_{ij} u_j(t, L_j) = \sum_{j=1}^N m_{ij} e^{-\int_{t-L_j+a_j}^{t-L_j+b_j} \alpha_j(s) ds} u_j(t - L_j, 0)$$

- Set $x(t) = (u_i(t, 0))_{i \in [\![1, N]\!]}$. Then x satisfies the difference equation

$$x(t) = \sum_{k=1}^N A_k(t) x(t - L_k)$$

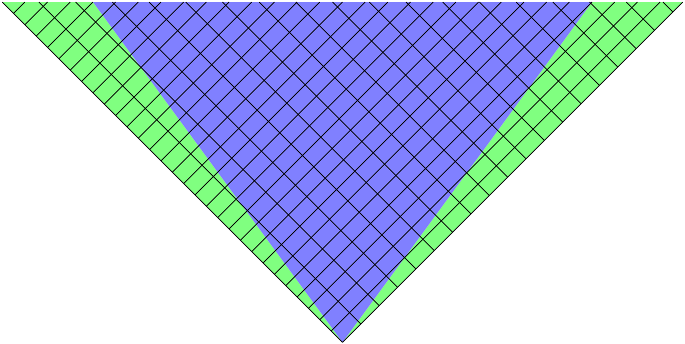
with

$$A_k(t) = \left(\delta_{jk} m_{ij} e^{-\int_{t-L_j+a_j}^{t-L_j+b_j} \alpha_j(s) ds}\right)_{i,j \in [\![1, N]\!]}$$

- It suffices to show that such difference equation is $(\widehat{\Xi}, L)$-exponentially stable. We study the behavior of the coefficients $\Xi_{\mathbf{n},t}^{L,A}$ as $\left|\mathbf{n}\right|_1 \to +\infty$.

Application to a transport system
Technique of the proof

Decomposition of the set $\mathbb{N}^N$.



Application to a transport system

Technique of the proof

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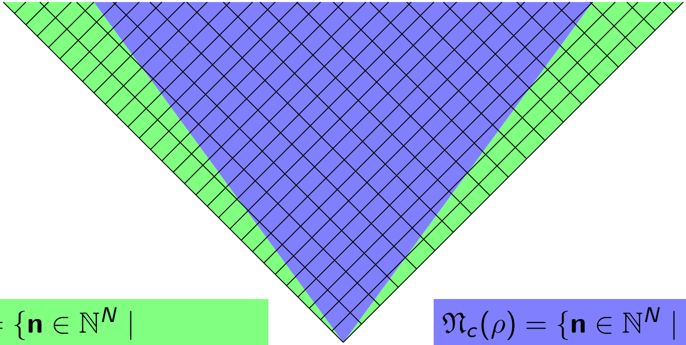

$\mathfrak{N}_b(\rho) = \{\mathbf{n} \in \mathbb{N}^N \mid \exists k \in [\![1, N]\!] \text{ s.t. } n_k \leq \rho \left|\mathbf{n}\right|_1\}$

$\mathfrak{N}_c(\rho) = \{\mathbf{n} \in \mathbb{N}^N \mid n_k > \rho \left|\mathbf{n}\right|_1, \forall k \in [\![1, N]\!]\}$

Application to a transport system

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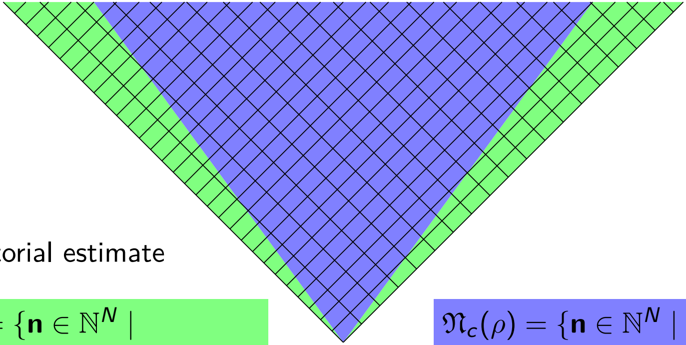

combinatorial estimate

$\mathfrak{N}_b(\rho) = \{\mathbf{n} \in \mathbb{N}^N \mid \exists k \in [\![1, N]\!] \text{ s.t. } n_k \leq \rho \left|\mathbf{n}\right|_1\}$

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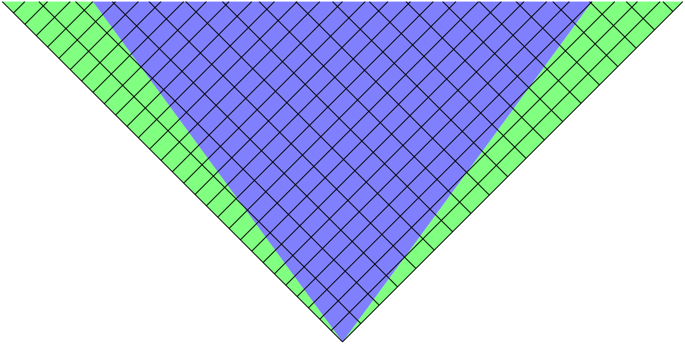
Application to a transport system
Technique of the proof

In $\mathfrak{N}_c(\rho)$: "box lemma"



Application to a transport system

Technique of the proof

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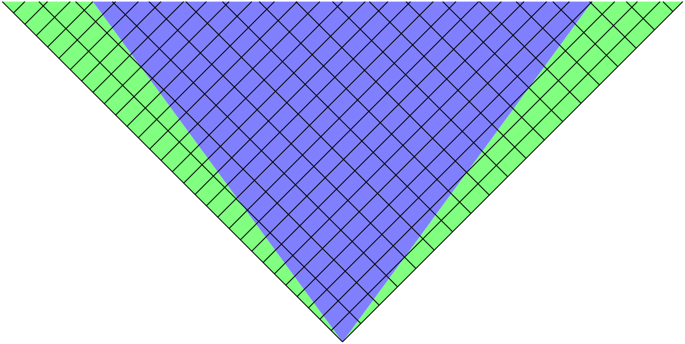
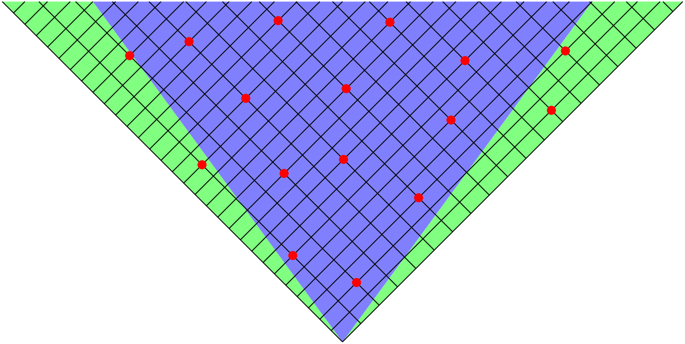


$$e^{-\int_{t-L\cdot\mathbf{n}+a_k}^{t-L\cdot\mathbf{n}+b_k}\alpha_k(s)ds}$$

Application to a transport system

Technique of the proof

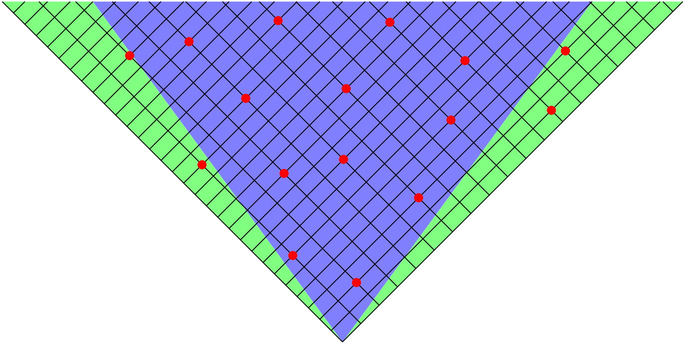
In $\mathfrak{N}_c(\rho)$: "box lemma"



$$e^{-\int_{t-L\cdot\mathbf{n}+a_k}^{t-L\cdot\mathbf{n}+b_k} \alpha_k(s)ds} \leq \eta$$

Application to a transport system
Technique of the proof

In $\mathfrak{N}_c(\rho)$: "box lemma"

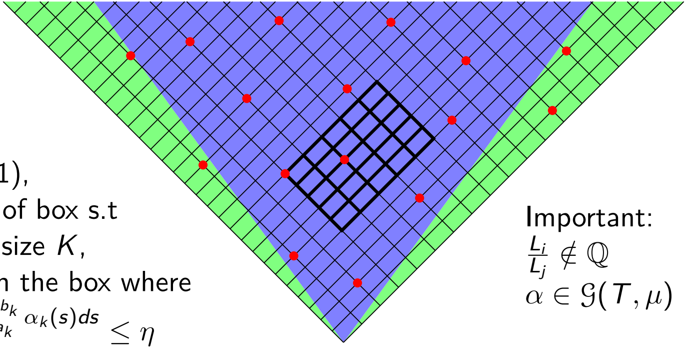


Find $\eta \in (0, 1)$ such that $e^{-\int_{t-L\cdot\mathbf{n}+a_k}^{t-L\cdot\mathbf{n}+b_k} \alpha_k(s) ds} \leq \eta$ "often enough"

Application to a transport system

Technique of the proof

In $\mathfrak{N}_c(\rho)$: "box lemma"



$\exists \eta \in (0, 1)$,
$\exists$ size K of box s.t
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Important:
$\frac{L_i}{L_j} \notin \mathbb{Q}$
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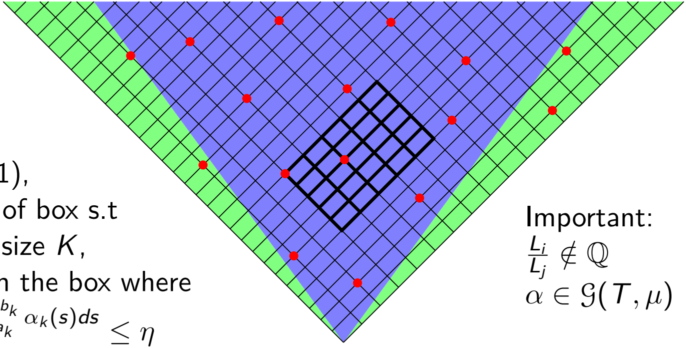
Application to a transport system

Technique of the proof

In $\mathfrak{N}_c(\rho)$: "box lemma"



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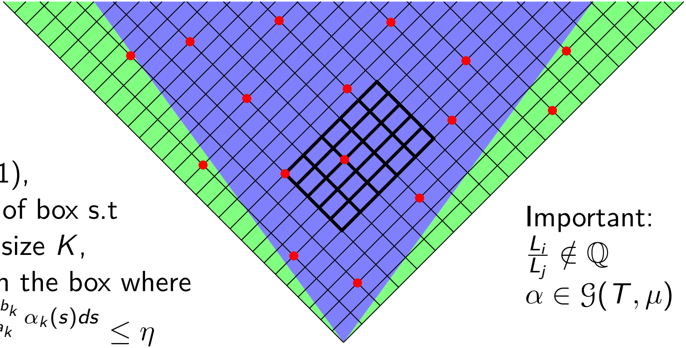

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We say that Σ_{contr} is relatively controllable in time $T > 0$ if, for every $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$ and $x_1 \in \mathbb{C}^d$, there exists $u : [0, T] \to \mathbb{C}^m$ such that the unique solution x of Σ_{contr} with initial condition x_0 and control u satisfies $x(T) = x_1$.

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By linearity, solution with initial condition x_0 and control u is the sum of this formula with the previous one.

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Relative controllability criterion

Theorem (M.; 2016)

The following statements are equivalent:

- Σ_{contr} *is relatively controllable in time* T*;*
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Theorem (M.; 2016)

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Theorem (M.; 2016)

- *If Σ_{contr} is relatively controllable in some time $T > 0$, then it is also relatively controllable in time $T = (d-1)\Lambda_{\max}$.*
- *If $\Lambda_1, \dots, \Lambda_N$ are rationally independent, then Σ_{contr} is relatively controllable in some time $T > 0$ if and only if*

$$\operatorname{Span}\left\{\Xi_{\mathbf{n}}^{A} B e_j \mid \mathbf{n} \in \mathbb{N}^N,\ \lvert\mathbf{n}\rvert_1 \leq d-1,\ j \in [\![1, m]\!]\right\} = \mathbb{C}^d.$$