Stabilization of a string network: optimal decay rates and numerical implementations

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Stability of nonconservative systems
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Outline

- I. Introduction
- II. Theoretical results (*)
- III. Numerical results (*)
- IV. Semi-Lagrangian implementation (new)
- V. Conclusion

(*) M. Jellouli, M. Mehrenberger, *Optimal decay rates for the stabilization of a string network*, Comptes Rendus Mathematiques, 2014.

The case of one string

- Vibrating elastic string fixed at end point

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0, \text{ on } (0,\ell), \; u(t,\ell) = 0$$

- Dissipation condition $\partial_x u(t,0) = \alpha \partial_t u(t,0)$ at origin
- Then the energy of the system satisfies

$$E(t) \le C \exp(-\gamma_\alpha t), \; \gamma = \frac{1}{\ell} \log \left| \frac{1+\alpha}{1-\alpha} \right|$$

- If $\alpha = 1$, then $E(t) = 0$, for $t \ge 2\ell$.

The case of one string

- See V. Komornik, *The Multiplier Method*
- Explicit solution via d'Alembert formula
- Exponential stabilization for $\alpha \neq 1$
- Controllability for $\alpha = 1$ (stabilization in finite time)

Question: what happens for a network of strings?

Some related results

- Star-shaped with stabilization at the common node or tree-shaped with stabilization at the root (Ammari, Jellouli, Khenissi, 2004/5)

  no exponential stability in the non degenerate case
- Pointwise stabilisation of a string viewed as 2 star shaped strings
  - (Ammari, Henrot, Tucsnak, 2001) Dirichlet/Neumann
    - exponential stabilization is obtained if and only if the lengths satisfy $\frac{\ell_1}{\ell_1+\ell_2} = \frac{p}{q}$, with p and q odd numbers
    - best decay rate, fixing total length $\ell_1 + \ell_2$ obtained for equal lengths $\ell_1 = \ell_2 = \ell$
    - best decay rate is $\gamma = \frac{\ln(3)}{\ell}$
  - (Nicaise, Valein, 2010) Dirichlet/Dirichlet, degenerate case
    - Energy limit E_∞ is identified
    - exponential decrease of energy to E_∞
- New results of L. Rosier and coauthors concerning *finite time stabilization*.
- New result of M. Jellouli, 2015: characterization of best decay rates with spectral analysis

A tree-shaped configuration

- $N \geq 3$ vibrating elastic strings

$$\partial_t^2 u_j(t,x) - \partial_x^2 u_j(t,x) = 0,\ t > 0,\ x \in (0,\ell_j),\ j = 1,\dots,N$$

- Continuity: $u_1(t,\ell_1) = u_j(t,0),\ j = 2,\dots,N$
- Dissipation at the root of the tree

$$\partial_x u_1(t,0) = \alpha \partial_t u_1(t,0),\ t \geq 0\ (\alpha > 0)$$

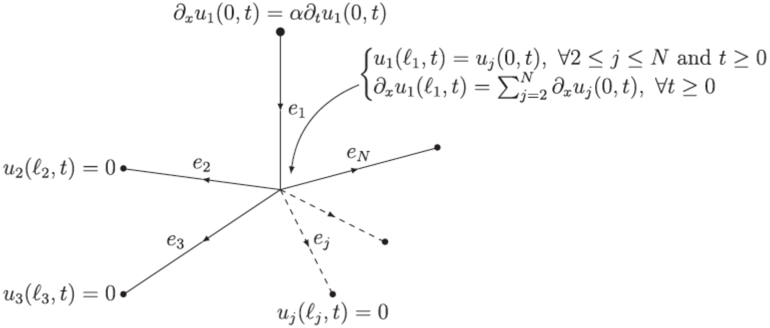
- Dirichlet at other exterior nodes $u_j(t,\ell_j) = 0,\ t \geq 0,\ j = 2,\dots,N$
- Transfert condition

$$\partial_x u_1(t,\ell_1) = \sum_{j=2}^{N} \partial_x u_j(t,0),\ t \geq 0,$$

- Initial condition

$$u_j(0,x) = a_j(x),\ \ \partial_t u_j(0,x) = b_j(x),\ x \in [0,\ell_j],\ j = 1,\dots,N.$$

Picture [1]



[1] borrowed from `Jellouli, 2015`

Questions

- What is the energy limit E_∞?
- Do we have exponential decrease to E_∞?
- What is the best decay rate?
- Can we have stabilization in finite time?
- How do numerical schemes behave?

In the sequel, we suppose that all the lengths are equal

$$\ell_j = \ell,\ j = 1, \dots, N.$$

The case $N = 2$ can be recasted to $N = 1$ with length $\ell_1 + \ell_2$.

Energy and energy limit

The energy is defined as

$$E(t) = \frac{1}{2}\sum_{j=1}^{N} \left\|\partial_x u_j(t)\right\|^2 + \frac{1}{2}\sum_{j=1}^{N} \left\|\partial_t u_j(t)\right\|^2,$$

with $\| \ \|$ for the norm of $L^2(0,\ell)$

Theorem

The energy limit is given by

$$E_\infty = \frac{1}{2(N-1)}\sum_{j=2}^{N}\sum_{k=j+1}^{N}\left(\left\|a'_k - a'_j\right\|^2 + \left\|b_k - b_j\right\|^2\right).$$

$E_\infty = 0$, for $a'_j = a'_2,\ b_j = b_2,\ j = 2, \ldots, N$.

Exponential decrease to E_∞ and best decay rate

Theorem

If $\alpha \neq \alpha_0$,

$$0 \leq E(t) - E_\infty \leq C_{\alpha,N}\left(\|a_1'\|^2 + \|b_1\|^2 + \left\| \sum_{j=2}^{N} a_j' \right\|^2 + \left\| \sum_{j=2}^{N} b_j \right\|^2 \right) e^{-\gamma t}$$

If $\alpha = \alpha_0$,

$$0 \leq E(t) - E_\infty \leq C_{\alpha,N}\left(\|a_1'\|^2 + \|b_1\|^2 + \left\| \sum_{j=2}^{N} a_j' \right\|^2 + \left\| \sum_{j=2}^{N} b_j \right\|^2 \right) t^2 e^{-\gamma_0 t},$$

where $\gamma_0 = \frac{1}{\ell} \log(1 + 2\frac{\sqrt{N-1}+1}{N-2})$ *and* $\gamma < \gamma_0$ *for* $\alpha \neq \alpha_0 = \frac{2\sqrt{N-1}}{N}$.

Remarks

- No stabilization in finite time, except for special initial data
- Proof based on operators of type τ (Ammari, Jellouli, 2007/10)
- Explicit computations based on d'Alembert formula
- Difficult to make the computations when lengths are different
- Star-shaped configuration should be feasible with same approach
- Explicit values permit to check the code
- Fixing total length L, we have $\ell = L/N$ and
  - $L\gamma_0$ is the worst for $N = 4$ (value is $4\log(2+\sqrt{3}) \simeq 5.2678$)
  - $L\gamma_{\alpha=1}$ is the worst for $N = \infty$ (value is 2)

Semi-discretization

- Classical finite difference space semi-discretization
- Discrete energy is decreasing; what is the limit?

Proposition

For $\phi \in \mathbb{R}^{2M}$*, and numerical solution* $V(t) = \exp(tA)\phi$*, we can write* $\phi = \phi_1 + \phi_2$ *where* ϕ_1 *belongs to the space* Λ_1 *of eigenvectors relative to the eigenvalues* λ *of A, with* $\Re(\lambda) = 0$ *and* ϕ_2 *belongs to the space* Λ_2 *of generalized eigenvectors relative to the eigenvalues* λ *of A, with* $\Re(\lambda) < 0$*. We then have*

$$E_{h,\phi,\infty} := \lim_{t\to\infty} E_{h,\phi}(t) = E_{h,\phi_1}(0),$$

and this value does not depend on $\alpha > 0$*.*

Numerical results

- We consider $d = 3$ strings of length $\ell = 1$
- N is now the number of points in each string
- initial condition non zero on string 2
  - $u_2^0(x) = \sin^2(\pi x)$
  - $u_2^0(x) = \sin^2(8\pi x)$

exact/semi-discrete time evolution of $E(t) - E_\infty$

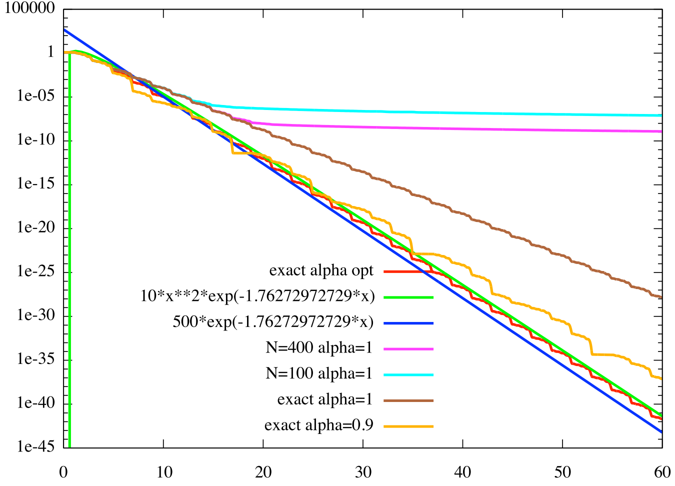


Figure: $\alpha \in \{0.9, 1, \alpha_{\rm opt} \simeq 0.9428\}$

exact/semi-discrete time evolution of $E(t) - E_\infty$

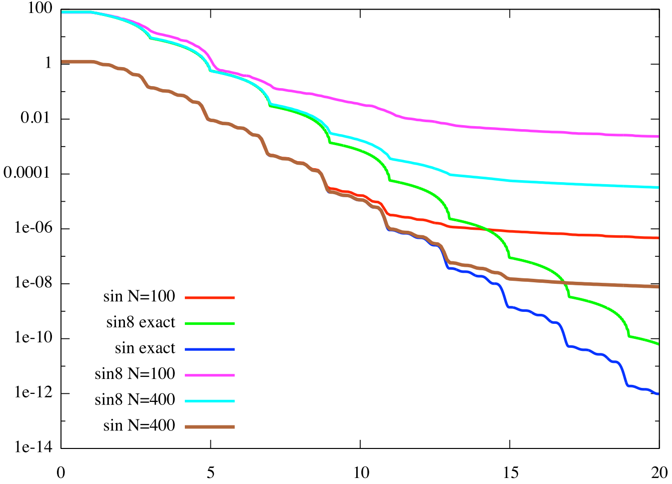
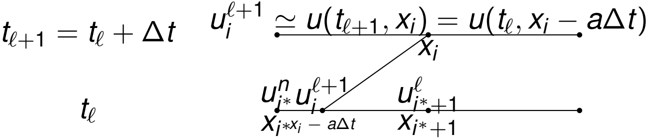


Figure: different initial conditions, $\alpha = \alpha_{\mathrm{opt}}$ }

Moving to a semi-Lagrangian scheme

The case of 1d constant advection

$$\partial_t u + a\partial_x u = 0, \; u = u(t,x)$$



- Characteristics are exact
- Lagrange interpolation :
  - Degree 1 (linear): x_{i^*}, x_{i^*+1}
  - Degree 3 (cubic): $x_{i^*-1}, x_{i^*}, x_{i^*+1}, x_{i^*+2}$
- Cubic splines interpolation
- Hermite interpolation :
  - f'_{i^*} : $x_{i^*-2}, x_{i^*-1}, x_{i^*}, x_{i^*+1}, x_{i^*+2}$

The case of Lagrange interpolation

Some known results

- L^2 stability `Strang, 1962`
- $L^q,\ q \geq 1$ stability for odd degree `Després, 2009`
- The scheme is equivalent to a Lagrange Galerkin scheme (`Pironneau, 1982`) for odd degree ≤ 13 `Ferretti, 2010`
  $\Rightarrow$ other proof of L^2stability

About SL-LG equivalence

Ferretti, 2010

- SL and LG are equivalent for the $1d$ constant advection, if we can find a function ϕ such that

$$\int_{\mathbb{R}} \phi(\eta + y)\phi(y)dy = \psi(y) \qquad \textbf{auto-correlation integral}$$

- ψ describes the Semi Lagrangian (SL) scheme
- ϕ describes the Lagrange Galerkin (LG) scheme
- In Fourier

$$\hat{\psi}(\omega) = \left|\hat{\phi}(\omega)\right|^2$$

- Example: for degree 3, we have

$$\hat{\psi}(\omega) = \frac{8(6+\omega^2)\sin(\omega/2)^4}{3\omega^4} \in \mathbb{R}^+$$

A direct proof of SL-LG equivalence

Algebraic form of the Fourier transform valid for *arbitrary* odd degree [2] (conjectured in Ferretti, 2010)

- Aim: prove that $S(\omega) = \int_0^1 \sum_{\ell=-d}^{d+1} L_\ell(x) \exp\left(i\,(\ell - x)\,\omega\right) dx \in \mathbb{R}^+$
- Compact formula for the derivative Boyer/Després's lecture notes
  $S'(\omega) = (-1)^d \frac{2^{2d+1}}{(2d+1)!} \sin^{2d+1}\left(\frac{\omega}{2}\right) \sigma(\omega)$
- **Integration by parts** for the factor $\sigma(\omega) = \int_0^1 \cos\left(\left(x - \frac{1}{2}\right)\omega\right) w\,(x)\, dx,\;\; w(x) = \prod_{j=-d}^{d+1}(x - j)$
- Recognise the primitive thanks to relation

$$w^{(2k+1)}(0) = -\frac{d+1}{2k+2} w^{(2k+2)}(0),\;\; k = 0, \ldots, d.$$

- Final explicit form

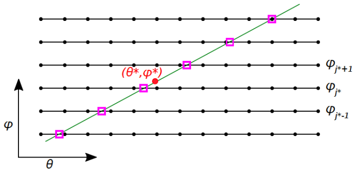
$$S(\omega) = (-1)^d \frac{2^{2d+1}}{(2d+1)!} \sin^{2d+2}\left(\frac{\omega}{2}\right) \sum_{k=0}^{d} \frac{w^{(2k+2)}(0)}{k+1} \frac{(-1)^k}{\omega^{2k+2}}$$

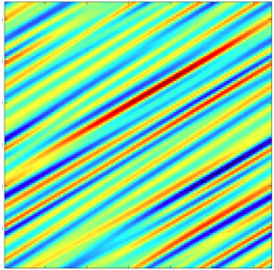
$\Rightarrow$ New proof of L^2 stability of SL scheme for constant advection

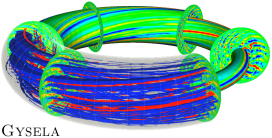
[2] Ferretti-M, WIP

Other proof from aligned interpolation[3]

Interpolation along a fixed oblic direction







GYSELA

$\Rightarrow$ Reconstruction of the values necessary by interpolation in θ

$\Rightarrow$ Reconstruction in the aligned direction

[3] Latu-Güclü-M-Ottaviani-Sonnendrücker, submitted

Other proof from aligned interpolation

- We first treat the case of $\lambda = \frac{b_\theta N_\theta}{b_\varphi N_\varphi}$ rational
  - *2d* symbol writes as a convex combination of 1*d* symbols in aligned direction
    - coefficients are discrete Fourier transform
    - Discrete Fourier transform is real
    - Discrete Fourier transform is **nonnegative**
- Case of λ real by density

Design of a semi-Lagrangian scheme for a string network

- d'Alembert formula is exact but solution can become complex
- finite difference scheme subject to CFL condition
- semi-Lagrangian method is the discrete analog of d'Alembert
- treatment of boundary
- link with Inverse Lax-Wendroff method [4]

[4] Tan-Shu, 2010; Chang-Filbet, 2013

Rewriting of the wave equation

Considering the wave equation

$$\partial_t^2 u - c^2 \partial_x^2 u = 0,$$

we can rewrite it as [5]

$$\begin{cases} \partial_t u + c\partial_x p = 0 \\ \partial_t p + c\partial_x u = 0. \end{cases}$$

We set

$$\begin{aligned} w^+ &= (u+p)/2 \\ w^- &= (u-p)/2 \end{aligned}$$

so that

$$\begin{cases} \partial_t w^+ + c\partial_x w^+ = 0 \\ \partial_t w^- - c\partial_x w^- = 0 \end{cases}$$

[5] see e.g. `Del Pino-Jourdren, 2006`

Initial condition

From

$$\begin{cases} u(t=0) = a_j \\ \partial_t u(t=0) = b_j \end{cases}$$

we get

$$\begin{cases} w_j^+(0,x) = \frac{1}{2}a_j(x) - \frac{1}{2c}\int_0^x b_j(y)dy - k_j \\ w_j^-(0,x) = \frac{1}{2}a_j(x) + \frac{1}{2c}\int_0^x b_j(y)dy + k_j \end{cases}$$

where k_j is an arbitrary constant.
The energy is then given by

$$E(t) = \frac{1}{2}\sum_{j=1}^{N}\int_0^{\ell_j} \left|\partial_x w^+ + \partial_x w^-\right|^2 + c^2\left|\partial_x w^+ - \partial_x w^-\right|^2$$

Boundary condition at root

At root, we have

$$\partial_x u = \alpha \partial_t u.$$

Note that

$$c\partial_x u = c\partial_x w^- + c\partial_x w^+ = \partial_t w^- - \partial_t w^+$$

and

$$\alpha c \partial_t u = \alpha c \partial_t w^- + \alpha c \partial_t w^+,$$

so that

$$\partial_t w^+ = \frac{1-\alpha c}{1+\alpha c}\partial_t w^-.$$

Integrating, we get

$$w^+(t+\Delta t, 0) = w^+(t,0) + \frac{1-\alpha c}{1+\alpha c}\left(w^-(t, c\Delta t) - w^-(t,0)\right).$$

Boundary condition at root

This leads to

$$w_1^+(t+\Delta t, x_k) = w_1^+(t+\Delta t - x_k/c_1 + x_k/c_1, x_k) = w_1^+(t+\Delta t - x_k/c_1, 0)$$
$$= w_1^+(t,0) + \frac{1-\alpha c_1}{1+\alpha c_1}\left(w_1^-(t, c_1\Delta t - x_k) - w_1^-(t,0)\right),$$

when $x_k < c_1\Delta t$, while

$$w_1^+(t+\Delta t, x_k) = w_1^+(t, x_k - c_1\Delta t),$$

otherwise.
Note that we have a *weak* restriction on time step

$$c_1\Delta t \le \ell_1.$$

Boundary condition at other exterior nodes

At other exterior nodes, we have $\partial_t u = 0$
This leads to $\partial_t w^+ + \partial_t w^- = 0$
By integration, we get

$$w^-(t+\Delta t,\ell) = w^-(t,\ell) + w^+(t,\ell) - w^+(t,\ell - c\Delta t)$$

For $j = 2,\ldots,N$, this leads to

$$\begin{aligned} & w_j^-(t+\Delta t, x_k) = w_j^-(t+\Delta t - (\ell_j - x_k)/c_j + (\ell_j - x_k)/c_j, x_k) \\ & = w_j^-(t+\Delta t - (\ell_j - x_k)/c_j, \ell_j) \\ & = w_j^-(t,\ell_j) + w_j^+(t,\ell_j) - w_j^+(t,\ell_j - c_j\Delta t + \ell_j - x_k) \end{aligned}$$

when $\ell_j - x_k < c_j \Delta t$, while

$$w_j^-(t+\Delta t, x_k) = w_j^-(t, x_k + c_j\Delta t)$$

otherwise. We have again the constraint $c_j \Delta t \leq \ell_j$.

Boundary condition at interior node

Suppose for simplicity that $N = 3$. We have

$$\partial_x u_1(\ell_1) = \partial_x u_2(0) + \partial_x u_3(0).$$

It rewrites

$$\partial_x w_1^+(\ell_1) + \partial_x w_1^-(\ell_1) = \partial_x w_2^+(0) + \partial_x w_2^-(0) + \partial_x w_3^+(0) + \partial_x w_3^-(0)$$

That is

$$\begin{array}{l} -\frac{1}{c_1}\partial_t w_1^+(\ell_1) + \frac{1}{c_1}\partial_t w_1^-(\ell_1) \\ = -\frac{1}{c_2}\partial_t w_2^+(0) + \frac{1}{c_2}\partial_t w_2^-(0) \\ -\frac{1}{c_3}\partial_t w_3^+(0) + \frac{1}{c_3}\partial_t w_3^-(0) \end{array}$$

$$\begin{array}{l} -\frac{1}{c_1} w_1^+(t+\Delta t, \ell_1) + \frac{1}{c_1} w_1^+(t, \ell_1) + \frac{1}{c_1} w_1^-(t+\Delta t, \ell_1) - \frac{1}{c_1} w_1^-(t, \ell_1) \\ = -\frac{1}{c_2} w_2^+(t+\Delta t, 0) + \frac{1}{c_2} w_2^+(t, 0) + \frac{1}{c_2} w_2^-(t+\Delta t, 0) - \frac{1}{c_2} w_2^-(t, 0) \\ -\frac{1}{c_3} w_3^+(t+\Delta t, 0) + \frac{1}{c_3} w_3^+(t, 0) + \frac{1}{c_3} w_3^-(t+\Delta t, 0) - \frac{1}{c_3} w_3^-(t, 0) \end{array}$$

Boundary condition at interior node

$$
\begin{aligned}
&-\tfrac{1}{c_1}w_1^+(t+\Delta t,\ell_1)+\tfrac{1}{c_1}w_1^+(t,\ell_1)+\tfrac{1}{c_1}w_1^-(t+\Delta t,\ell_1)-\tfrac{1}{c_1}w_1^-(t,\ell_1)\\
&=-\tfrac{1}{c_2}w_2^+(t+\Delta t,0)+\tfrac{1}{c_2}w_2^+(t,0)+\tfrac{1}{c_2}w_2^-(t+\Delta t,0)-\tfrac{1}{c_2}w_2^-(t,0)\\
&-\tfrac{1}{c_3}w_3^+(t+\Delta t,0)+\tfrac{1}{c_3}w_3^+(t,0)+\tfrac{1}{c_3}w_3^-(t+\Delta t,0)-\tfrac{1}{c_3}w_3^-(t,0)
\end{aligned}
$$

$$
\begin{aligned}
&-\tfrac{1}{c_1}w_1^+(t,\ell_1-c_1\Delta t)+\tfrac{1}{c_1}w_1^+(t,\ell_1)+\tfrac{1}{c_1}w_1^-(t+\Delta t,\ell_1)-\tfrac{1}{c_1}w_1^-(t,\ell_1)\\
&=-\tfrac{1}{c_2}w_2^+(t+\Delta t,0)+\tfrac{1}{c_2}w_2^+(t,0)+\tfrac{1}{c_2}w_2^-(t,c_2\Delta t)-\tfrac{1}{c_2}w_2^-(t,0)\\
&-\tfrac{1}{c_3}w_3^+(t+\Delta t,0)+\tfrac{1}{c_3}w_3^+(t,0)+\tfrac{1}{c_3}w_3^-(t,c_3\Delta t)-\tfrac{1}{c_3}w_3^-(t,0)
\end{aligned}
$$

This leads to the first relation

$$
\begin{aligned}
&\tfrac{1}{c_1}w_1^-(t+\Delta t,\ell_1)+\tfrac{1}{c_2}w_2^+(t+\Delta t,0)+\tfrac{1}{c_3}w_3^+(t+\Delta t,0)\\
&=\tfrac{1}{c_1}w_1^-(t,\ell_1)+\tfrac{1}{c_2}w_2^+(t,0)+\tfrac{1}{c_3}w_3^+(t,0)\\
&+\tfrac{1}{c_1}w_1^+(t,\ell_1-c_1\Delta t)-\tfrac{1}{c_1}w_1^+(t,\ell_1)\\
&+\tfrac{1}{c_2}w_2^-(t,c_2\Delta t)-\tfrac{1}{c_2}w_2^-(t,0)\\
&+\tfrac{1}{c_3}w_3^-(t,c_3\Delta t)-\tfrac{1}{c_3}w_3^-(t,0)
\end{aligned}
$$

Boundary condition at interior node

We have also the conditions for $j = 2, \ldots, N$

$$\partial_t u_1(\ell_1) = \partial_t u_j(0).$$

This leads to

$$w_1^+(t+\Delta t, \ell_1) + w_1^-(t+\Delta t, \ell_1) = w_1^+(t, \ell_1) + w_1^-(t, \ell_1)$$
$$+ w_j^+(t+\Delta t, 0) + w_j^-(t+\Delta t, 0) - w_j^+(t, 0) - w_j^-(t, 0)$$

that is

$$w_1^+(t, \ell_1 - c_1\Delta t) + w_1^-(t+\Delta t, \ell_1) = w_1^+(t, \ell_1) + w_1^-(t, \ell_1)$$
$$+ w_j^+(t+\Delta t, 0) + w_j^-(t, c_j\Delta t) - w_j^+(t, 0) - w_j^-(t, 0)$$

$$w_1^-(t+\Delta t, \ell_1) - w_j^+(t+\Delta t, 0) = w_1^-(t, \ell_1) - w_j^+(t, 0)$$
$$+ w_1^+(t, \ell_1) - w_1^+(t, \ell_1 - c_1\Delta t) + w_j^-(t, c_j\Delta t) - w_j^-(t, 0)$$

Boundary condition at interior node

Matrix to solve, $A = L^{-1}U$, for example for $N = 4$

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1/c_1 & 1/c_2 & 1/c_3 & 1/c_4 \end{pmatrix}$$

with

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1/c_2 & 1/c_3 & -\sum_{i=1}^{N-1} 1/c_i & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \sum_{i=1}^{N} 1/c_i \end{pmatrix}$$

Boundary condition at interior node

Now, we use as before

$$w_1^-(t+\Delta t, x_k) = \begin{cases} w_1^-(t + (x_k + c_1\Delta t - \ell_1)/c_1, \ell_1), \ x_k + c_1\Delta t > \ell_1 \\ w_1^-(t, x_k + c_1\Delta t), \ x_k + c_1\Delta t \leq \ell_1 \end{cases}$$
$$w_j^+(t+\Delta t, x_k) = \begin{cases} w_j^+(t - (x_k - c_j\Delta t)/c_j, 0), \ x_k - c_j\Delta t < 0 \\ w_j^+(t, x_k - c_j\Delta t), \ x_k - c_j\Delta t \geq 0 \end{cases}$$

So, if $\ell_1 - x_k < c_1\Delta t$, we need to compute for $\delta t = \Delta t - (\ell_1 - x_k)/c_1$

$$\begin{array}{l} w_1^+(t, \ell_1 - c_1\delta t) \\ w_j^-(t, c_j\delta t), \ j = 2, \ldots, N \end{array}$$

If $x_k < c_2\Delta t$, we need to compute the same quantities for $\delta t = \Delta t - x_k/c_2$, and so on until N instead of 2.

Interpolation

We use **Hermite** interpolation [6], instead of Lagrange interpolation.
Fix $N_1 \in \mathbb{N}^*$ and $\varepsilon > 0$
We divide $(0, \ell_1)$ into N_1 cells $dx = \ell_1 / N_1$

- First cell

$$0, \varepsilon, (1-\varepsilon)dx, (1+\varepsilon)dx$$

- Cell $\ell \in \{2, \ldots, N_1 - 1\}$

$$(\ell - 1 - \varepsilon)dx, (\ell - 1 + \varepsilon)dx, (\ell - \varepsilon)dx, (\ell + \varepsilon)dx$$

- Cell number N_1

$$(N_1 - 1 - \varepsilon)dx, (N_1 - 1 + \varepsilon)dx, (N_1 - \varepsilon)dx, N_1 dx = \ell_1$$

[6] JET schemes `Seibold-Rozales-Navé, 2012`

Interpolation

On a cell (a, b), we have for cell $\ell \in \{2, \ldots, N_1 - 1\}$

$$x_1 = a - \varepsilon dx,\ x_2 = a + \varepsilon dx,$$
$$x_3 = b - \varepsilon dx,\ x_4 = b + \varepsilon dx$$

and use Hermite interpolation using points

$$f_a = (f(x_1) + f(x_2))/2,\ f'_a = (f(x_2) - f(x_1))/(2\varepsilon),$$
$$f_b = (f(x_3) + f(x_4))/2,\ f'_b = (f(x_4) - f(x_3))/(2\varepsilon)$$

For $\alpha \in (0, 1)$

$$f_{(1-\alpha)a+\alpha b} = (1-\alpha)^2(2\alpha+1)f_a + \alpha^2(3-2\alpha)f_b + \alpha(1-\alpha)^2 f'_a - \alpha^2(1-\alpha)f'_b$$

Adaption for first and last cell

Interpolation

Letting $\varepsilon \to 0$, we return to classical Hermite interpolation, using

$$f'_{(1-\alpha)a+\alpha b} = 6\alpha(1-\alpha)(f_b - f_a) + (3\alpha-1)(\alpha-1)f'_a + \alpha(3\alpha-2)f'_b$$

We use a uniform mesh and have to locate the cells and displacement inside the cell

We consider first the first string with $x_k = (k-1)\frac{\ell_1}{N_1}, k = 1, \dots, N_1 + 1$.

- $-c_1 N_1 \Delta t/\ell_1 = -k_1 + \alpha_1,\ k_1 \in \mathbb{Z},\ 0 \le \alpha_1 < 1$.
- If $\Delta t = 0$, we take $k_1 = 1,\ \alpha_1 = 1$. Then, we can suppose $k_1 \in \{1, \dots, N_1\}$.
- $x_{k_1+1} - c_1 \Delta t$: displacement α_1 in cell 1 ...
- $x_{N_1+1} - c_1 \Delta t$: displacement α_1 in cell $N_1 + 1 - k_1$
- $c_1 N_1 \Delta t/\ell_1 = k_1 - 1 + 1 - \alpha_1$
- $c_1 \Delta t - x_1$: displacement $1 - \alpha_1$ in cell k_1 ...
- $c_1 \Delta t - x_{k_1}$: displacement $1 - \alpha_1$ in cell 1

Interpolation

- $x_1 + c_1 \Delta t$: displacement $1 - \alpha_1$ in cell k_1 ...
- $x_{N_1 - k_1 + 1} + c_1 \Delta t$: displacement $1 - \alpha_1$ in cell N_1
- $2\ell_1 - x_k - c_1 \Delta t = \frac{\ell_1}{N_1}(2N_1 - k - k_1 + 1 + \alpha_1)$
- $2\ell_1 - x_{N_1 - k_1 + 2} - c_1 \Delta t$: displacement α_1 in cell N_1 ...
- $2\ell_1 - x_{N_1 + 1} - c_1 \Delta t$: displacement α_1 in cell $N_1 + 1 - k_1$

Finally, we need to compute

- w_1^+ at each cell with shift α_1
- w_1^- at each cell with shift $1 - \alpha_1$

This is not enough

Interpolation

We write

$$\Delta t = \frac{\ell_j}{c_j N_j}\left(k_j - 1 + 1 - \alpha_j\right),\ j = 1, \ldots, N.$$

We define $k_{j,j'}(s),\ \alpha_{j,j'}(s)$ by

$$\frac{\ell_j}{c_j N_j}\left(k_{j,j'}(s) - 1 + 1 - \alpha_{j,j'}(s)\right) = \frac{\ell_{j'}}{c_{j'} N_{j'}}\left(s - 1 + 1 - \alpha_{j'}\right)$$

We need here

$$k_{j,1}(1), \ldots, k_{j,1}(k_1) = k_j,\ \alpha_{j,1}(1), \ldots, \alpha_{j,1}(k_1) = \alpha_j$$

- $c_j(\Delta t - (\ell_1 - x_{N_1-k_1+2})/c_1)$: displacement $\alpha_{j,1}(1)$ in cell $k_{j,1}(1)$ ...
- $c_j(\Delta t - (\ell_1 - x_{N_1+1})/c_1)$: displacement $\alpha_{j,1}(k_1)$ in cell $k_{j,1}(k_1)$

Interpolation

Now, we consider the other strings $j = 2, \ldots, N$
Now, $x_k = (k-1)\frac{\ell_j}{N_j}, k = 1, \ldots, N_j + 1$.

- $x_{k_j+1} - c_j\Delta t$: displacement α_j in cell 1 $\ldots$
- $x_{N_j+1} - c_j\Delta t$: displacement α_j in cell $N_j + 1 - k_j$
- $c_j\Delta t - x_1$: displacement $1 - \alpha_j$ in cell k_j $\ldots$
- $c_j\Delta t - x_{k_j}$: displacement $1 - \alpha_j$ in cell 1
- $x_1 + c_j\Delta t$: displacement $1 - \alpha_j$ in cell k_j $\ldots$
- $x_{N_j-k_j+1} + c_j\Delta t$: displacement $1 - \alpha_j$ in cell N_j
- $2\ell_j - x_{N_j-k_j+2} - c_j\Delta t$: displacement α_j in cell N_j $\ldots$
- $2\ell_j - x_{N_j+1} - c_j\Delta t$: displacement α_j in cell $N_j + 1 - k_j$

This is again not enough

Interpolation

We have for $j' \neq j$

$$c_{j'}(\Delta t - \frac{x_k}{c_j}) = \frac{c_j' \ell_j}{c_j N_j}(k_j - k + 1 - 1 + 1 - \alpha_j)$$
$$= \frac{\ell_{j'}}{N_{j'}}(k_{j'j}(k_j - k + 1) - 1 + 1 - \alpha_{j'j}(k_j - k + 1))$$

- $c_{j'}(\Delta t - x_1/c_j)$: displacement $\alpha_{j',j}(k_j)$ in cell $k_{j',j}(k_j)$ ...
- $c_{j'}(\Delta t - x_{k_j}/c_j)$: displacement $\alpha_{j',j}(1)$ in cell $k_{j',j}(1)$

It remains to treat the case

$$\ell_1 - c_1(\Delta t - \frac{x_k}{c_j}) = \ell_1 - \frac{\ell_1}{N_1}(k_{1j}(k_j - k + 1) - 1 + 1 - \alpha_{1j}(k_j - k + 1))$$
$$= \frac{\ell_1}{N_1}(N_1 - k_{1j}(k_j - k + 1) + \alpha_{1j}(k_j - k + 1))$$

- $\ell_1 - c_1(\Delta t - \frac{x_1}{c_j})$: displacement $\alpha_{1,j}(k_j)$ in cell $N_1 + 1 - k_{1,j}(k_j)$ ...
- $\ell_1 - c_1(\Delta t - \frac{x_{k_j}}{c_j})$: displacement $\alpha_{1,j}(1)$ in cell $N_1 + 1 - k_{1,j}(1)$

Special discretization

We suppose that all the lengths and speeds are rational.
Taking $M \in \mathbb{N}^*$ large enough such that $N_j = M\ell_j/c_j \in \mathbb{N}^*$ and $k_j = k_1$, with

$$k_1 \in \{1, \ldots, \min_{k=1,\ldots,N} M\ell_k/c_k\}.$$

We take

$$\Delta t = \frac{k_1}{M} = \frac{\ell_j k_j}{c_j N_j},\ j = 1, \ldots, N.$$

and there is no interpolation: we get the exact solution.

- Low computational cost (and acceleration possible, when no inteprolation is needed)
- Low memory (treat outflow first)
- Good candidate for numerical optimization of parameters for more complex cases as those currently solved (as optimal decay rates)

Link with Lattice Boltzmann method

A BGK model Let $\varepsilon > 0$ and $c \geq 0$. We consider the model

$$\partial_t f + v\partial_x f = -\frac{1}{\varepsilon}(f - M(f)).$$

where $f = f(t, x, v)$, with $v \in \{\pm 1, 0\}$ and the *maxwellian* $M(f)$ is defined through the relations:

$$\begin{cases} \rho(f) = \int f dv = f(\cdot,\cdot,-1) + f(\cdot,\cdot,0) + f(\cdot,\cdot,1), \\ \rho(f)u(f) = \int fv dv = f(\cdot,\cdot,1) - f(\cdot,\cdot,-1), \end{cases} \tag{1}$$

and

$$\begin{cases} \rho(f) = \int M(f) dv = M(f)(\cdot,\cdot,-1) + M(f)(\cdot,\cdot,0) + M(f)(\cdot,\cdot,1), \\ \rho(f)u(f) = \int M(f) v dv = M(f)(\cdot,\cdot,1) - M(f)(\cdot,\cdot,-1), \\ \rho(f)(u(f)^2 + c^2) = \int M(f) v^2 dv = M(f)(\cdot,\cdot,1) + M(f)(\cdot,\cdot,-1). \end{cases} \tag{2}$$

Conclusion/Perspectives

- Conclusion
  - Stabilization for a tree-shaped network with equal lengths
  - Numerical results are coherent with theoretical ones
  - Design of a semi-Lagrangian scheme
- Perspectives
  - Validation of the scheme w.r.t. theoretical results
  - Numerical analysis of the scheme
  - Numerical study for other configurations