Pulse control for heat equations

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The heat equation

Ω open bounded set in $\mathbb{R}^d$,

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ u(\cdot,0) = u_0 & \in L^2(\Omega) \ , \end{cases}$$

$$\boxed{\|u(\cdot,t)\|_{L^2(\Omega)} \leq e^{-\lambda_1 t} \|u_0\|_{L^2(\Omega)}}$$

Goal: To have a better decay rate by a feedback control.

Constraints:

* To act on $\omega \times (n+1)\,T$ for any $n \in \mathbb{N}$;
* Sensors are on $\omega_1 \times \left(n+\frac{1}{2}\right)T$ for any $n \in \mathbb{N}$.

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } \Omega \times (0,+\infty) \setminus \{(\mathbb{N}+1)T\} \ , \\ y = 0 & \text{on } \partial\Omega \times (0,+\infty) \ , \\ y(\cdot,0) = y_0 & \text{in } \Omega \ , \\ \forall n \in \mathbb{N} & \\ y(\cdot,(n+1)\,T) = y(\cdot,(n+1)\,T-) + \chi_\omega \mathcal{F}\left(y_{|\omega_1 \times \{(n+\frac{1}{2})T\}}\right) \ , \end{cases}$$

$$\|y(\cdot,t)\|_{L^2(\Omega)} \leq Ce^{-\gamma t}\,\|y_0\|_{L^2(\Omega)} \text{ and } \|\mathcal{F}\| \leq Ce^{C\gamma}$$

Impulse Stabilization: Aim of fast decay

$$\forall \gamma > 0 \quad \exists \mathcal{F} : L^2(\omega_1) \to L^2(\omega) \quad \forall y_0 \in L^2(\Omega)$$

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } \Omega \times (0,+\infty) \setminus \{(\mathbb{N}+1)T\} \ , \\ y = 0 & \text{on } \partial\Omega \times (0,+\infty) \ , \\ y(\cdot,0) = y_0 & \text{in } \Omega \ , \\ \forall n \in \mathbb{N} & \\ y(\cdot,(n+1)T) = y(\cdot,(n+1)T-) + \chi_\omega \mathcal{F}\left(y_{|\omega_1 \times \{(n+\frac{1}{2})T\}}\right) & , \end{cases}$$

$$\|y(\cdot,t)\|_{L^2(\Omega)} \leq Ce^{-\gamma t} \|y_0\|_{L^2(\Omega)} \text{ and } \|\mathcal{F}\| \leq Ce^{C\gamma}$$

Impulse Stabilization: Main result

$$\forall \gamma > 0 \quad \exists \mathcal{F} : L^2(\omega_1) \to L^2(\omega) \quad \forall y_0 \in L^2(\Omega)$$

$$\begin{cases} \partial_t y - \Delta y = \displaystyle\sum_{n\in\mathbb{N}} \delta_{(n+1)T} \otimes \chi_\omega \mathcal{F}\left(y_{|\omega_1 \times \{(n+\frac{1}{2})T\}} \right), \\ y = 0 \quad \text{on } \partial\Omega \times (0, +\infty), \\ y(\cdot, 0) = y_0 \quad \text{in } \Omega, \end{cases}$$

$$\|y(\cdot,t)\|_{L^2(\Omega)} \le Ce^{-\gamma t} \|y_0\|_{L^2(\Omega)} \quad \text{and} \quad \|\mathcal{F}\| \le Ce^{C\gamma}$$

Strategy

To have in mind that for the model wave equation

$$\boxed{\textit{Observability estimate} \quad \Longleftrightarrow \quad \textit{Controllability}}$$

$$\boxed{\Longleftrightarrow \quad \textit{Stabilization} \quad \Longrightarrow \quad \textit{Inverse Source}}$$

References

Observability : Fursikov-Imanuvilov, Lebeau-Robbiano,...
Measurable sets : Apraiz-Escauriaza-Wang-Zhang, Vessella
Backward estimate : Bardos-Tartar

Collaboration with Claude Bardos, Gengsheng Wang, Yashan Xu

Observation at one point in time

Ω smooth bounded domain in $\mathbb{R}^d$, ω non-empty open subset of Ω

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ u(\cdot,0) = u_0 & \in L^2(\Omega) \ , \end{cases}$$

$$OPT : \quad \boxed{\left\| u(\cdot,T) \right\|_{L^2(\Omega)} \le e^{C\left(1+\frac{1}{T}\right)} \left\| u(\cdot,T) \right\|_{L^2(\omega)}^{\theta} \left\| u_0 \right\|_{L^2(\Omega)}^{1-\theta}}$$

Plan of the talk

Proof of Observation estimates at one point in time :

* by Lebeau-Robbiano estimate of Sum of Eigenfunctions;

* by Log Convexity method and Frequency Function.

Applications :

* Impulse Stabilization for heat equations where input and output have different localizations;

* Observation of solutions of the Neutron Transport Equation in the diffusion limit.

Remark 1

$$\boxed{OPT \quad \Longrightarrow \quad Observability\ estimate}$$

$$\boxed{OPT \quad \Longleftrightarrow \quad LR}$$

Lebeau-Robbiano sum of eigenfunctions: (Reference: Jerison-Lebeau (1996))

$$\begin{cases} -\Delta e_i = \lambda_i e_i \quad \text{in } \Omega \ , \\ e_i \in H_0^1(\Omega) \ , \quad 0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N \leq \cdots \ . \end{cases}$$

$$LR: \quad \boxed{\sum_{\lambda_i \leq \lambda} |a_i|^2 \leq C e^{C\sqrt{\lambda}} \int_\omega \left| \sum_{\lambda_i \leq \lambda} a_i e_i(x) \right|^2 dx}$$

OPT⇒LR :Take $u_0 = \sum_{\lambda_i \leq \lambda} a_i e^{\lambda_i T} e_i$ and minimize w.r.t. T.

LR⇒OPT $\forall\theta$:Take $u_0 = \sum_{\lambda_i \leq \lambda} a_i e_i + \sum_{\lambda_i > \lambda} a_i e_i$ and use decay

$$\left\| e^{T\Delta} \sum_{\lambda_i > \lambda} a_i e_i \right\|_{L^2(\Omega)} \leq e^{-\lambda T} \left\| u_0 \right\|_{L^2(\Omega)}$$ and optimize w.r.t. λ.

Log convexity

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq \|u(\cdot,T)\|_{L^2(\Omega)}^{t/T} \|u_0\|_{L^2(\Omega)}^{1-t/T}$$

because $t \mapsto \int_\Omega |u(x,t)|^2 dx$ is log convex. Now $t \mapsto \Phi(t) = \int_\Omega |u(x,t)|^2 G_\hbar(x,t)\, dx$ is log convex in the sense

$$G_\hbar(x,t) = \frac{1}{(T-t+\hbar)^{d/2}} e^{-\frac{|x-x_0|^2}{4(T-t+\hbar)}} \text{ and } N(t) = \frac{\int_\Omega |\nabla u(x,t)|^2 G_\hbar(x,t) dx}{\Phi(t)}$$

$$\frac{1}{2}\Phi' + N\Phi = 0 \text{ and } N' \leq \frac{1}{T-t+\hbar} N$$

$$\Longrightarrow \Phi(t_2) \leq \Phi(t_3)^{\frac{1}{1+\alpha}} \Phi(t_1)^{\frac{\alpha}{1+\alpha}}$$

Here, $\hbar$ is a small parameter.

Remark 2 : Advantages of Log convexity

1. One can replace $\partial_t - \Delta$ by $\partial_t - \text{div}(a(x,t)\nabla\cdot)$
2. One can replace $\partial_t - \Delta$ by $\partial_t - \Delta + a(x,t)$
3. Sum of eigenfunction estimate for Schroedinger operator $-\Delta + a(x)$ for λ large

$$\sum_{\lambda_i \leq \lambda} |a_i|^2 \leq C e^{C\left(1+\|a\|_{L^\infty(\Omega)}^{2/3}+\sqrt{\lambda}\right)} \int_\omega \left| \sum_{\lambda_i \leq \lambda} a_i e_i(x) \right|^2 dx$$

Remark 3 : Explicit constant wrt geometry

When Ω is convex bounded in $\mathbb{R}^d$ and $\omega = \{|x - x_0| < R\} \subset \Omega$, then for any $u_0 \in L^2(\Omega), T > 0$, $(a_i)_{i\geq 1} \in \mathbb{R}$, $\lambda \geq 1$, $\epsilon \in (0,1)$,

$$\left\|e^{T\Delta}u_0\right\|_{L^2(\Omega)} \leq \frac{1}{R^{d+\epsilon(d-2)}} e^{\frac{C}{T}\frac{1}{R^{6\epsilon}}} \int_0^T \left\|e^{t\Delta}u_0\right\|_{L^2(\omega)} dt$$

$$\left\|\sum_{\lambda_i\leq\lambda} a_i e_i\right\|_{L^2(\Omega)} \leq \frac{1}{R^{d(1+\epsilon)}} e^{C\sqrt{\lambda}\frac{1}{R^{2\epsilon}}} \left\|\sum_{\lambda_i\leq\lambda} a_i e_i\right\|_{L^2(\omega)}$$

Application 1 : Pulse Control and Cost

$$\forall T_1 < T_2 < T_3 \quad \forall \varepsilon > 0 \quad \forall y_e \in L^2(\Omega) \quad \exists f \in L^2(\Omega)$$

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } \Omega \times (T_1, T_3) \backslash \{T_2\} \ , \\ y = 0 & \text{on } \partial\Omega \times (T_1, T_3) \ , \\ y(\cdot, T_1) = y_e & \text{in } \Omega \ , \\ y(\cdot, T_2) = y(\cdot, T_2 -) + \chi_\omega f & \text{in } \Omega \ , \\ y_g = y(\cdot, T_3) & \text{in } \Omega \ , \end{cases}$$

$$\|f\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon^\gamma} \|y_e\|_{L^2(\Omega)} \quad \text{and} \quad \|y_g\|_{L^2(\Omega)} \leq \varepsilon \|y_e\|_{L^2(\Omega)}$$

$$\text{Minimize } J(\varphi_0) = \frac{cost}{2} \int_\omega |\varphi(x, T_3 + T_1 - T_2)|^2 dx + \frac{\varepsilon}{2} \int_\Omega |\varphi_0(x)|^2 dx - \int_\Omega y_e(x) \varphi(x, T_3) dx$$

Application 2 : Pulse Control of eigenfunctions

$$\forall T_1 < T_2 < T_3 \quad \forall \varepsilon > 0 \quad \forall i \quad \exists f_i \in L^2(\Omega)$$

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } \Omega \times (T_1, T_3) \backslash \{T_2\} \ , \\ y = 0 & \text{on } \partial\Omega \times (T_1, T_3) \ , \\ y(\cdot, T_1) = e_i & \text{in } \Omega \ , \\ y(\cdot, T_2) = y(\cdot, T_2-) + \chi_\omega f_i & \text{in } \Omega \ , \\ y_{gi} = y(\cdot, T_3) & \text{in } \Omega \ , \end{cases}$$

$$\|f_i\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon^\gamma} \text{ and } \|y_{gi}\|_{L^2(\Omega)} \leq \varepsilon$$

Application 3 : Pulse Control of a sum of eigenfunctions

$$\forall T_1 < T_2 < T_3 \quad \forall \varepsilon > 0 \quad \forall i \quad \forall a_i \quad \exists f_i \in L^2(\Omega)$$

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } \Omega \times (T_1, T_3) \setminus \{T_2\} \ , \\ y = 0 & \text{on } \partial\Omega \times (T_1, T_3) \ , \\ y(\cdot, T_1) = \sum_{i=1}^{M} a_i e_i & \text{in } \Omega \ , \\ y(\cdot, T_2) = y(\cdot, T_2 -) + \chi_\omega \sum_{i=1}^{M} a_i f_i & \text{in } \Omega \ , \\ \sum_{i=1}^{M} a_i y_{gi} = y(\cdot, T_3) & \text{in } \Omega \ , \end{cases}$$

$$\left\| \sum_{i=1}^{M} a_i f_i \right\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon^\gamma} \left\| (a_i) \right\|_{\ell^2} \text{ and } \left\| y(\cdot, T_3) \right\|_{L^2(\Omega)} \leq \varepsilon \left\| (a_i) \right\|_{\ell^2}$$

Application 4 : Inverse Pulse Source problem (i.e. find a_i)

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (T_1, T_3) \ , \\ u = 0 & \text{on } \partial\Omega \times (T_1, T_3) \ , \\ u(\cdot, T_1) = \sum_{i \geq 1} a_i e_i & \text{is unknown but in } L^2(\Omega) \ . \end{cases}$$

We know u on $\omega \times \{\tau\}$, then $\forall i \quad \exists \widehat{f_i} \in L^2(\Omega)$ with $\left\|\widehat{f_i}\right\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon^\gamma}$

$$\left| a_i + e^{(T_3 - T_1)\lambda_i} \int_\omega \widehat{f_i}(x)\, u(x, \tau)\, dx \right| \leq \varepsilon e^{(T_3 - T_1)\lambda_i} \left\|(a_i)\right\|_{\ell^2}$$

Choose $\widehat{f_i}$ the impulse control at time $T_2 = T_3 + T_1 - \tau$ associated to the inital data the eigenfunction $y_e = e_i$

Application 5 : Output Fast Impulse Stabilization

$$\forall \gamma > 0 \quad \exists \mathcal{F}_\gamma : L^2(\omega_1) \to L^2(\omega) \quad \forall y_0 \in L^2(\Omega)$$

$$\left\{ \begin{array}{rl} \partial_t y - \Delta y = 0 & \text{in } \Omega \times (0, +\infty) \backslash \{(\mathbb{N}+1)T\} \ , \\ y = 0 & \text{on } \partial\Omega \times (0, +\infty) \ , \\ y(\cdot, 0) = y_0 & \text{in } \Omega \ , \\ \forall n \in \mathbb{N} & \\ y(\cdot, (n+1)T) = y(\cdot, (n+1)T-) + \chi_\omega \mathcal{F}_\gamma \left(y_{|\omega_1 \times \{(n+\frac{1}{2})T\}} \right) & , \end{array} \right.$$

$$\|y(\cdot, t)\|_{L^2(\Omega)} \leq C e^{-\gamma t} \|y_0\|_{L^2(\Omega)} \quad \text{and} \quad \|\mathcal{F}_\gamma\| \leq C e^{C\gamma}$$

Ideas of proof

Set $L_n = nT + \frac{1}{4}T$. Suppose $y(\cdot, L_n) = \sum_{i\geq 1} a_i e_i$. Since we know y on $\omega_1 \times \{nT + \frac{1}{2}T\}$, we get by Inverse Pulse Source

$$\left| a_i + e^{(T_3 - T_1)\lambda_i} \int_{\omega_1} \widehat{f}_i(x)\, y\left(x, \left(n + \tfrac{1}{2}\right) T\right) dx \right| \leq \varepsilon e^{\frac{T}{2}\lambda_i} \left\| (a_i) \right\|_{\ell^2}$$

with $T_1 = L_n$ and $T_3 = nT + \frac{3}{4}T$

Next we use Pulse Control at time $(n+1)T$ with

$\mathcal{F}(z) = \sum_{i=1}^{M} -e^{\frac{T}{2}\lambda_i} \int_{\omega_1} \widehat{f}_i(x)\, z(x)\, dx\ f_{i|\omega}$ in order

$$\left\| y(\cdot, L_{n+1}) \right\|_{L^2(\Omega)} \leq e^{-\gamma T} \left\| y(\cdot, L_n) \right\|_{L^2(\Omega)}$$

Radiative transfert equation

$$\Gamma_- = \left\{(x,v) \in \partial\Omega \times \mathbb{S}^{d-1}; v \cdot n_x < 0\right\},$$

$\mathsf{S} f = f - \frac{1}{\left|\mathbb{S}^{d-1}\right|} \int_{\mathbb{S}^{d-1}} f dv = f - \langle f \rangle$ or $\mathsf{S} = \frac{-1}{d-1} \Delta_{\mathbb{S}^{d-1}}$

$$\begin{cases} \partial_t f_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla f_\varepsilon + \frac{1}{\varepsilon^2} \mathsf{S} f_\varepsilon = 0 & \text{in } \Omega \times \mathbb{S}^{d-1} \times (0,T) \ , \\ f_\varepsilon = 0 & \text{in } \Gamma_- \times (0,T) \ , \\ f_\varepsilon(x,v,0) = f^{in}(x,v) & \text{for } (x,v) \in \Omega \times \mathbb{S}^{d-1} \ . \end{cases}$$

$$\langle f_\varepsilon \rangle \to u \text{ when } \varepsilon \to 0$$

$$\begin{cases} \partial_t u - \frac{1}{d} \Delta u = 0 & \text{in } \Omega \times (0,T) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ u(\cdot, 0) = \left\langle f^{in} \right\rangle & \text{in } \Omega \ . \end{cases}$$

Approximation diffusion

Hilbert expansion : $\forall f^{in} \in H_0^1(\Omega)$ with $f^{in} = \langle f^{in} \rangle$

$$\left\| f_\varepsilon(\cdot,\cdot,T) - u(\cdot,T) \right\|_{L^2(\Omega\times\mathbb{S}^{d-1})} \leq \sqrt{\varepsilon} C \left\| f^{in} \right\|_{H_0^1(\Omega)}$$

Moment method : $\forall p > 2 \quad \forall f^{in} \in L^{2p}\left(\Omega \times \mathbb{S}^{d-1}\right)$

$$\left\| \langle f_\varepsilon \rangle (\cdot,T) - u(\cdot,T) \right\|_{H^{-1}(\Omega)} \leq \varepsilon^{\frac{1}{2p}} C(T,p) \left\| f^{in} \right\|_{L^{2p}\left(\Omega\times\mathbb{S}^{d-1}\right)}$$

Main result

$\forall p > 2 \quad \forall f^{in} \in L^{2p}\left(\Omega \times \mathbb{S}^{d-1}\right)$

let $\quad \sigma\left(f^{in}\right) = c\left[1 + \frac{1}{T} + T\frac{\left\|\left\langle f^{in}\right\rangle\right\|^2_{L^2(\Omega)}}{\|\langle f^{in}\rangle\|^2_{H^{-1}(\Omega)}}\right]$

and $\quad R\left(f^{in}\right) = C_p\left(1 + T^{\frac{p-1}{2p}}\right)\frac{\left\|f^{in}\right\|_{L^{2p}(\Omega\times\mathbb{S}^{d-1})}}{\|\langle f^{in}\rangle\|_{L^2(\Omega)}} e^{\sigma\left(f^{in}\right)}$

then

$$\left[1 - \varepsilon^{\frac{1}{2p}} R\left(f^{in}\right)\right]\left\|\left\langle f^{in}\right\rangle\right\|_{L^2(\Omega)} \leq e^{\sigma\left(f^{in}\right)}\left\|\langle f_\varepsilon\rangle(\cdot, T)\right\|_{L^2(\omega)}$$

Thank You