Singularly perturbed hyperbolic systems

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Stability of non-conservative systems
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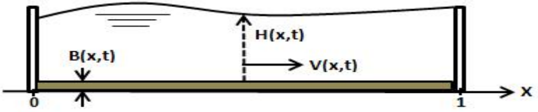
Motivations: Saint-Venant–Exner system

- Open channel problem



Prismatic open channel

◇ rectangular cross-section
◇ losses are negligible

$$
\begin{aligned}
H_t + VH_x + HV_x &= 0, \\
V_t + VV_x + gH_x + gB_x &= 0, \quad x \in [0,1], \quad t \in [0,+\infty), \\
B_t + aV^2 V_x &= 0.
\end{aligned}
$$

$H(x,t)$ - water level ; $V(x,t)$ - water velocity ; $B(x,t)$ - bathymetry ; g - gravity constant; a - constant parameter on sediment porosity.

The linearized system with respect to a space constant steady-state $(H^\star, V^\star, B^\star)$ is

$$\begin{pmatrix} h \\ v \\ b \end{pmatrix}_t + \begin{pmatrix} V^\star & H^\star & 0 \\ g & V^\star & g \\ 0 & aV^{*2} & 0 \end{pmatrix} \begin{pmatrix} h \\ v \\ b \end{pmatrix}_x = 0.$$

Performing a change of variable, we get a hyperbolic system

$$W_t + \Lambda W_x = 0,$$

with

$$W_k = \frac{\Big((V^\star - \lambda_i)(V^\star - \lambda_j) + gH^\star\Big) h + H^\star \lambda_k v + gH^\star b}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}, \quad k \neq i \neq j \in \{1, 2, 3\},$$

and $\Lambda = diag(\lambda_1, \lambda_2, \lambda_3)$, see [Diagne, Bastin, Coron; 2012]

- λ_1 and λ_3: velocity of the water flow
- λ_2: velocity of the sediment motion

$$\lambda_2 << |\lambda_1|, \quad \lambda_2 << \lambda_3.$$

Defining $\varepsilon = \frac{\lambda_2}{\lambda_3}$ and $\tilde{t} = \lambda_2 t$, and a change of spatial variable $W_1'(1-x,t) = W_1(x,t)$, we obtain a singularly perturbed hyperbolic system as follows

$$W_{\tilde{t}} + \Lambda' W_x = 0,$$

with $\Lambda' = diag(\frac{|\lambda_1|}{\varepsilon\lambda_3}, 1, \frac{1}{\varepsilon})$.

- Boundary conditions depend on the control

What happens if ε is small in terms of the stability?

Could we design boundary controllers taking into account the two-scale dynamics?

Since ε is small, the Courant Friedrichs Lewy condition asks that $\frac{\Delta x}{\Delta t}$ is very small.

Is it possible to scale the equations of the so-called singularly perturbed system and to develop specific control theory.

Outline

1. Singularly perturbed systems in finite-dimensional systems
   linear ODE versus nonlinear ODE

   Pedagogical purpose
2. Singularly perturbed hyperbolic systems

   linear PDE but counter-example of the intuitive idea
3. Stability of singularly perturbed hyperbolic systems
4. Approximation result

   Tikhonov theorem for linear hyperbolic systems
5. Further results on coupled ODE-PDE

   only partial results extra work is (still) needed
6. Boundary control of the Saint-Venant–Exner system

   application on some numerical simulations
7. Conclusion

$$\begin{cases} \dot{y}(t) &= Ay(t) + Bz(t) \\ \varepsilon\dot{z}(t) &= Cy(t) + Dz(t) \end{cases}$$

with $y(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^m$, $\varepsilon > 0$ small
Formally we have by letting $\varepsilon = 0$ in z-equation

$$z = -D^{-1}Cy$$

By replacing z by $-D^{-1}Cy$ in the y equation, we get the following reduced system

$$\dot{\overline{y}} = A_r\overline{y}$$

with $A_r = A - BD^{-1}C$. By using the following change of variables $\overline{z}(t/\varepsilon) = z(t) + D^{-1}Cy(t)$ we get: $\varepsilon\dot{\overline{z}} = D\overline{z} + \varepsilon D^{-1}C(Ay + Bz)$
Now using the following time-scale $\tau = t/\varepsilon$ and using (formally) $\varepsilon \to 0$, the boundary layer system is

$$\frac{d\overline{z}}{d\tau} = D\overline{z}$$

Stability of reduced system and of boundary layer systems implies the stability of the full system:

Proposition [Kokotović et al.; 1972]

If A_r and D have all eigenvalues in the (open) left-part of the plane, then there exists ε^* such that, for all $\varepsilon \in (0, \varepsilon^\star]$, the full system is exponentially stable.

Proof We write the dynamics into the coordinate $(y, \bar{z})$:

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ \bar{z}(\tau) \end{pmatrix} = \begin{pmatrix} Ay + Bz \\ \frac{1}{\varepsilon} D\bar{z} \end{pmatrix}$$

$$= \begin{pmatrix} A_r & B \\ 0 & \frac{1}{\varepsilon} D \end{pmatrix} \begin{pmatrix} y(t) \\ \bar{z}(\tau) \end{pmatrix}$$

and we conclude by letting ε sufficiently small ■

False for nonlinear ODEs

Stability of reduced and boundary layer systems

$\not\Rightarrow$ stability of the nonlinear ODE

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What about the approximation between the full system and the "small" systems?

Tikhonov theorem:

Proposition [Kokotović et al., 1986]

If A_r and D have all eigenvalues in the (open) left-part of the plane, then, given an initial condition, there exist $a > 0$ and ε^* such that, for all $t \geq 0$,

$$|y(t) - \overline{y}(t)| \leq a\varepsilon \quad (1)$$

$$|z(t) + D^{-1}C\overline{y}(t) - \overline{z}(t/\varepsilon)| \leq a\varepsilon \quad (2)$$

Sketch of proof of (1) and (2):

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Proof of (1): Recall that $\bar{z}(t/\varepsilon) = e^{Dt/\varepsilon}\bar{z}(0)$ and compute

$$\frac{d}{dt}(y - \bar{y}) = B\bar{z}$$

thus we have (1)

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Proof of (2): Easy computations give

$$\begin{aligned}
&\frac{d}{dt}(\bar{z}(t/\varepsilon) - z(t) - D^{-1}Cy(t)) \\
&= \frac{1}{\varepsilon}D\bar{z}(t/\varepsilon) - \frac{1}{\varepsilon}Cy(t) - \frac{1}{\varepsilon}Dz(t) - D^{-1}CAy(t) - D^{-1}CBz(t) \\
&= -\frac{1}{\varepsilon}D(\bar{z}(t/\varepsilon) - z(t) - D^{-1}Cy(t)) \\
&\quad -D^{-1}CAy(t) - D^{-1}CBz(t)
\end{aligned}$$

integrating and using $y(t)$, $z(t) \to 0$, we have (2)

The full system is given as follows

$$\begin{aligned} y_t(x,t) + \Lambda_1 y_x(x,t) &= 0, \quad y \in \mathbb{R}^n \\ \varepsilon z_t(x,t) + \Lambda_2 z_x(x,t) &= 0, \quad z \in \mathbb{R}^m \end{aligned} \tag{3}$$

where $\varepsilon > 0$ and Λ_1 and Λ_2 are diagonal positive, $x \in [0,1]$, $t \geqslant 0$.

The boundary conditions are

$$\begin{pmatrix} y(0,t) \\ z(0,t) \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} y(1,t) \\ z(1,t) \end{pmatrix}, \ t \in [0,+\infty), \tag{4}$$

with K_{11} in $\mathbb{R}^{n\times n}$, K_{12} in $\mathbb{R}^{n\times m}$, K_{21} in $\mathbb{R}^{m\times n}$, K_{22} in $\mathbb{R}^{m\times m}$.

The initial conditions are

$$\begin{pmatrix} y(x,0) \\ z(x,0) \end{pmatrix} = \begin{pmatrix} y^0(x) \\ z^0(x) \end{pmatrix}, \quad x \in [0,1].$$

Setting $\varepsilon = 0$ in the full system and assuming $(I_m - K_{22})$ invertible, we get formally

$$y_t(x,t) + \Lambda_1 y_x(x,t) = 0, \tag{5a}$$

$$z_x(x,t) = 0. \tag{5b}$$

Substituting (5b) into the full system's boundary conditions matrix, yields

$$\begin{aligned} z(.,t) &= (I_m - K_{22})^{-1} K_{21} y(1,t), \\ y(0,t) &= (K_{11} + K_{12}(I_m - K_{22})^{-1} K_{21}) y(1,t). \end{aligned}$$

The reduced subsystem is computed as

$$\bar{y}_t(x,t) + \Lambda_1 \bar{y}_x(x,t) = 0,\ x \in [0,1],\ t \in [0,+\infty), \tag{6}$$

with the boundary condition

$$\bar{y}(0,t) = K_r \bar{y}(1,t),\ t \in [0,+\infty), \tag{7}$$

where $K_r = K_{11} + K_{12}(I_m - K_{22})^{-1} K_{21}$.

The initial condition is as the same as the full system

$$\bar{y}(x,0) = y^0(x), \quad x \in [0,1].$$

Let us perform the following change of variable: $\bar{z}(x,t) = z(x,t) - (I_m - K_{22})^{-1} K_{21} y(1,t)$. Noting $\tau = t/\varepsilon$ and making $\varepsilon \to 0$, the boundary layer subsystem is

$$\bar{z}_\tau(x,\tau) + \Lambda_2 \bar{z}(x,\tau) = 0 \tag{8}$$

with the boundary condition

$$\bar{z}(0,\tau) = K_{22}\bar{z}(1,\tau)$$

and the initial condition

$$\bar{z}(x,0) = z_0(x) - (I_m - K_{22})^{-1} K_{21} y(1,0)$$

(short) review of the literature on the boundary stabilization of hyperbolic PDE

Many technics exist for one-scale linear hyperbolic system:

$$\begin{array}{c} \partial_t y + \Lambda \partial_x y = 0 \;, \quad x \in [0,1],\; t \geq 0 \\ y(0,t) = Ky(1,t) \;, \quad t \geq 0 \end{array} \tag{9}$$

There are sufficient conditions on K so that (9) is Locally Exponentially Stable in H^2, or in C^1...
[Coron, Bastin, d'Andréa-Novel; 08]
[Coron, Vazquez, Krstic, Bastin; 13]
[CP, Winkin, Bastin; 08]
Notation:

$$\begin{array}{c} \|K\| = \max\{|Ky|,\; y \in \mathbb{R}^n,\; |y| = 1\} \\ \rho(K) = \inf\{\|\Delta K \Delta^{-1}\|,\; \Delta \in \mathcal{D}_{n,+}\} \end{array}$$

[Coron *et al*; 08]: if $\rho(K) < 1$ then the system (9) is Exp. Stable in L^2-norm, and in H^2 norm
This sufficient condition is weaker that the one of [Li; 94].

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In other words

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If $\rho(K) < 1$ then the system (9) is exp. stable in L^2-norm that is $\exists\ \omega,\ C > 0$ such that for all $y_0 \in L^2(0,1)$,

$$\|y(.,t)\|_{L^2(0,1)} \leq Ce^{-\omega t}\|y_0\|_{L^2(0,1)}\ ,\ \forall t \geq 0.$$

Proof From $\rho(K) < 1$, there exists a diagonal positive definite matrix Δ such that $\|\Delta G \Delta^{-1}\| < 1$. Then, letting $Q = \Delta^2 \Lambda^{-1}$, we have

$$\Lambda Q - K^\top Q \Lambda K > 0 \tag{10}$$

Thus with a suitable $\mu > 0$, letting $V(y) = \int_0^1 e^{-\mu x} y(x)^\top Q y(x) dx$

$$\begin{aligned}\dot{V} &= -2\int_0^1 e^{-\mu x} y_x(x)^\top \Lambda^\top Q y(x) dx \\ &= -\mu \int_0^1 e^{-\mu x} y(x)^\top \Lambda^\top Q y(x) dx - [e^{-\mu x} y(x) Q \Lambda y(x)]_0^1\end{aligned}$$

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Remark It is also exp. stable in H^2 norm
that is $\exists\ \omega,\ C > 0$ such that for all $y_0 \in H^2(0,1)$ satisfying some compatibility conditions

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Proposition

$\rho(K) < 1 \implies$ the boundary layer and the reduced systems are both exp. stable in L^2 norm and in H^2

Proof

- Use some algebraic computations to show that $\rho(K_{22}) < 1$ and $\rho(K_r) < 1$
- Apply the previously recalled sufficient condition. ■

It is useless since we are more interesting in the converse implication

which is true for finite dimensional systems

but false in our case!!

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Exp. stability of the boundary layer system + exp. stability of the reduced system

$\not\Longrightarrow$

Exp. stability of the full system!

Indeed consider

$$\begin{array}{c} \partial_t y + \partial_x y = 0 \ , \quad x \in [0,1], \ t \geq 0 \\ \varepsilon \partial_t z + \partial_x z = 0 \ , \quad x \in [0,1], \ t \geq 0 \\ \begin{pmatrix} y(0,t) \\ z(0,t) \end{pmatrix} = \begin{pmatrix} 2.5 & -1 \\ 1 & 0.5 \end{pmatrix} \begin{pmatrix} y(1,t) \\ z(1,t) \end{pmatrix}, \quad t \geq 0 \end{array} \tag{11}$$

Recall: [Coron et al., 2008]: The condition $\rho(K) < 1$ is sufficient for exp. stability but also necessary for $n \leq 5$ for irrationally independent velocities.

We may check that $\rho(K) > 1$. Therefore, picking $\varepsilon \in \mathbb{R} \setminus \mathbb{Q}$, (11) is unstable.

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The reduced system

$$\begin{aligned} \bar{y}_t + \bar{y}_x &= 0 \\ \bar{y}(0,t) &= 0.5\bar{y}(1,t) \end{aligned}$$

and the boundary layer system

$$\begin{aligned} \bar{z}_\tau + \bar{z}_x &= 0 \\ \bar{z}(0,\tau) &= 0.5\bar{y}(1,\tau) \end{aligned}$$

are both exp. stable.
Therefore

Stability of subsystems $\not\Longrightarrow$ Stability of full system

What should be added?

To ease the computations, assume $y \in \mathbb{R}$, $z \in \mathbb{R}$ and $\Lambda_1 = \Lambda_2 = 1$.

Assumption #1

The reduced system (6) is exponentially stable in L^2-norm.

Assumption #2

The boundary-layer system (8) is exponentially stable in L^2-norm.

Assume moreover that

Assumption #3

Given $0 < d < 1$, $\mu > 0$ and $\nu > 0$ such that $e^{-\mu} > K_{11}^2$, $e^{-\mu} > \left(K_{11} + \frac{K_{12}K_{21}}{1-K_{22}}\right)^2$ and $e^{-\nu} > K_{22}^2$, assume

a) $(1-d)R_1 - dK_{21}^2 \geqslant 0$,

b) $dR_2 - (1-d)K_{12}^2 \geqslant 0$,

c) $((1-d)R_1 - dK_{21}^2)(dR_2 - (1-d)K_{12}^2) - ((1-d)R_3 + dR_4)^2 \geqslant 0$

where: $R_1 = e^{-\mu} - K_{11}^2$, $R_2 = e^{-\nu} - K_{22}^2$, $R_3 = K_{11}K_{12}$, $R_4 = K_{21}K_{22}$.

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The reduced system (6) is exponentially stable in L^2-norm.

Assumption #2

The boundary-layer system (8) is exponentially stable in L^2-norm.

Assume moreover that

Assumption #3

Given $0 < d < 1$, $\mu > 0$ and $\nu > 0$ such that $e^{-\mu} > K_{11}^2$, $e^{-\mu} > \left(K_{11} + \frac{K_{12}K_{21}}{1-K_{22}}\right)^2$ and $e^{-\nu} > K_{22}^2$, assume

a) $(1-d)R_1 - dK_{21}^2 \geqslant 0$,

b) $dR_2 - (1-d)K_{12}^2 \geqslant 0$,

c) $((1-d)R_1 - dK_{21}^2)(dR_2 - (1-d)K_{12}^2) - ((1-d)R_3 + dR_4)^2 \geqslant 0$

where: $R_1 = e^{-\mu} - K_{11}^2$, $R_2 = e^{-\nu} - K_{22}^2$, $R_3 = K_{11}K_{12}$, $R_4 = K_{21}K_{22}$.

What should be added?

To ease the computations, assume $y \in \mathbb{R}$, $z \in \mathbb{R}$ and $\Lambda_1 = \Lambda_2 = 1$.

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Assumption #2

The boundary-layer system (8) is exponentially stable in L^2-norm.

Assume moreover that

Assumption #3

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c) $\left((1-d)R_1 - dK_{21}^2\right)\left(dR_2 - (1-d)K_{12}^2\right) - ((1-d)R_3 + dR_4)^2 \geqslant 0$

where: $R_1 = e^{-\mu} - K_{11}^2$, $R_2 = e^{-\nu} - K_{22}^2$, $R_3 = K_{11}K_{12}$, $R_4 = K_{21}K_{22}$.

Sufficient for the exp. stability of the full system

Theorem [Tang, CP, Girard; 2013]

Under Assumptions #1, #2, and #3, there exists $\varepsilon^\star$ such that for all $0 < \varepsilon < \varepsilon^\star$, the full system is exp. stable in H^2-norm. Moreover it has the following Lyapunov function:

$$V(y,z) = (1-d)\int_0^1 e^{-\mu x}(y^2 + y_x^2 + y_{xx}^2)dx$$
$$+d\int_0^1 e^{-\nu x}\left(\left(z - \frac{K_{21}}{1-K_{22}}y(1)\right)^2 + \eta_1(\varepsilon)z_x^2 + \eta_2(\varepsilon)z_{xx}^2\right)$$

where η_1, η_2 are positive functions of ε.

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where η_1, η_2 are positive functions of ε.

Sketch of proof

First, let us decompose $V(y,z)$ as $V(y,z) = L_1 + L_2 + L_3$ with

$$
\begin{aligned}
L_1 &= (1-d)\int_0^1 e^{-\mu x} y^2 dx + d\int_0^1 e^{-\nu x}\left(z - \frac{K_{21}}{1-K_{22}} y(1)\right)^2 dx,\\
L_2 &= (1-d)\int_0^1 e^{-\mu x} y_x^2 dx + d\eta_1(\varepsilon)\int_0^1 e^{-\nu x} z_x^2 dx,\\
L_3 &= (1-d)\int_0^1 e^{-\mu x} y_{xx}^2 dx + d\eta_2(\varepsilon)\int_0^1 e^{-\nu x} z_{xx}^2 dx
\end{aligned}
$$

There are 4 steps in the proof:

- Estimation of $\dot{L}_1$
- Estimation of $\dot{L}_2$
- Estimation of $\dot{L}_3$
- Combining all computations

First use the dynamics and integrate by parts. We get

$$\dot{L}_1 = L_{11} + L_{12}$$

with

$$L_{11} = -(1-d)\left[e^{-\mu x} y^2\right]_{x=0}^{x=1} - \frac{d}{\varepsilon}\left[e^{-\nu x}\left(z - \frac{K_{21}}{1-K_{22}} y(1)\right)^2\right]_{x=0}^{x=1},$$

and

$$\begin{aligned} L_{12} = & -(1-d)\mu \int_0^1 e^{-\mu x} y^2 dx \\ & + \left(\frac{2dK_{21}}{1-K_{22}}\right) \int_0^1 e^{-\nu x}\left(z - \frac{K_{21}}{1-K_{22}} y(1)\right) y_x(1) dx \\ & - \frac{d}{\varepsilon}\nu \int_0^1 e^{-\nu x}\left(z - \frac{K_{21}}{1-K_{22}} y(1)\right)^2 dx. \end{aligned}$$

With the boundary conditions (4) and noting $z(1) = \left(z(1) - \frac{K_{21}}{1-K_{22}} y(1)\right) + \frac{K_{21}}{1-K_{22}} y(1)$, it follows

$$L_{11} = - \begin{pmatrix} y(1) \\ z(1) - \frac{K_{21}}{1-K_{22}} y(1) \end{pmatrix}^T M_{11} \begin{pmatrix} y(1) \\ z(1) - \frac{K_{21}}{1-K_{22}} y(1) \end{pmatrix}$$

with

$$M_{11} = \begin{pmatrix} (1-d) m_1 & -(1-d) K_2 \\ -(1-d) m_2 & \frac{d}{\varepsilon} R_2 - (1-d) K_{12}^2 \end{pmatrix},$$

where m_1, m_2 are some values and R_2 is defined in Assumption #3. Due to Assumptions #1 and #2, m_1 and R_2 are positive. Thus $L_{11} \leq 0$ as soon as $0 < \varepsilon \leq \varepsilon_1$ for a suitable ε_1.

Therefore $\boxed{\dot{L}_1 \leq L_{12}}$

Differentiating (3) with respect to x, we have

$$\begin{array}{rcl} y_{xt}(x,t) + y_{xx}(x,t) & = & 0, \\ \varepsilon z_{xt}(x,t) + z_{xx}(x,t) & = & 0, \end{array} \qquad (12)$$

with the boundary conditions

$$\begin{array}{rcl} y_x(0,t) & = & K_{11} y_x(1,t) + \frac{1}{\varepsilon} K_{12} z_x(1,t), \\ z_x(0,t) & = & \varepsilon K_{21} y_x(1,t) + K_{22} z_x(1,t). \end{array} \qquad (13)$$

Compute the time derivative of the second term L_2 along the solutions to (12) and (13)

$$\dot{L}_2 = L_{21} + L_{22}$$

with

$$\begin{array}{rcl} L_{21} & = & -(1-d)[e^{-\mu x} y_x^2]_{x=0}^{x=1} - \dfrac{d\eta_1(\varepsilon)}{\varepsilon}[e^{-\nu x} z_x^2]_{x=0}^{x=1}, \\ L_{22} & = & -(1-d)\mu \displaystyle\int_0^1 e^{-\mu x} y_x^2 dx - \dfrac{d\nu\eta_1(\varepsilon)}{\varepsilon} \displaystyle\int_0^1 e^{-\nu x} z_x^2 dx. \end{array}$$

Take $\eta_1(\varepsilon) = \frac{1}{\varepsilon}$, under the boundary conditions (13), it follows

$$L_{21} = -\begin{pmatrix} y_x(1) \\ z_x(1) \end{pmatrix}^T M_{21} \begin{pmatrix} y_x(1) \\ z_x(1) \end{pmatrix},$$

with:

$$M_{21} = \begin{pmatrix} (1-d)R_1 - dK_{21}^2 & -\frac{(1-d)R_3}{\varepsilon} - \frac{dR_4}{\varepsilon} \\ -\frac{(1-d)R_3}{\varepsilon} - \frac{dR_4}{\varepsilon} & \frac{dR_2 - (1-d)K_{12}^2}{\varepsilon^2} \end{pmatrix},$$

where R_1, R_2, R_3 and R_4 are defined in Assumption #3.
Assumption #3 a) and b) give that both terms on the diagonal are non-negative
Assumption #3 c) gives that the determinant is non-negative for all ε

(the choice of $\eta_1(\varepsilon)$ was crucial for that)

Therefore $\boxed{\dot{L}_2 \leq L_{22}}$

Step #3: Estimation of $\dot{L}_3$

Step 3: Differentiating (12) with respect to x, we have:

$$\begin{array}{rcl} y_{xxt}(x,t) + y_{xxx}(x,t) & = & 0, \\ \varepsilon z_{xxt}(x,t) + z_{xxx}(x,t) & = & 0, \end{array} \tag{14}$$

with boundary conditions:

$$\begin{array}{rcl} y_{xx}(0,t) & = & G_{11} y_{xx}(1,t) + \frac{1}{\varepsilon^2} K_{12} z_{xx}(1,t), \\ z_{xx}(0,t) & = & \varepsilon^2 K_{21} y_{xx}(1,t) + K_{22} z_{xx}(1,t). \end{array} \tag{15}$$

Compute the time derivative of the third term L_3 along the solutions to (14) and (15)

$$\dot{L}_3 = L_{31} + L_{32}$$

with

$$\begin{array}{rcl} L_{31} & = & -(1-d)[e^{-\mu x} y_{xx}^2]_{x=0}^{x=1} - \dfrac{d\eta_2(\varepsilon)}{\varepsilon}[e^{-\nu x} z_{xx}^2]_{x=0}^{x=1}, \\ L_{32} & = & -(1-d)\mu \displaystyle\int_0^1 e^{-\mu x} y_{xx}^2 dx - \dfrac{d\nu\eta_2(\varepsilon)}{\varepsilon} \displaystyle\int_0^1 e^{-\nu x} z_{xx}^2 dx. \end{array}$$

Take $\eta_2(\varepsilon) = \frac{1}{\varepsilon^3}$, under the boundary conditions (15), it follows

$$L_{31} = -\begin{pmatrix} y_{xx}(1) \\ \frac{z_{xx}(1)}{\varepsilon} \end{pmatrix}^T M_{11} \begin{pmatrix} y_{xx}(1) \\ \frac{z_{xx}(1)}{\varepsilon} \end{pmatrix}.$$

with the same matrix M_{11} as in Step #1. Recall that, for suitable $0 < \varepsilon \leq \varepsilon_1$, we have $M_{11} \geqslant 0$, thus L_{31} is non positive.

Therefore $\boxed{\dot{L}_3 \leq L_{32}}$

Step 4: We obtain that

$$\dot{L} \leq -(1-d)\mu \int_0^1 e^{-\mu x}(y^2 + y_x^2 + y_{xx}^2)dx$$
$$-\frac{d\nu}{\varepsilon}\int_0^1 e^{-\nu x}\left(\left(z - \frac{K_{21}}{1-K_{22}}y(1)\right)^2 + \frac{z_x^2}{\varepsilon} + \frac{z_{xx}^2}{\varepsilon^3}\right)$$
$$+\left(\frac{2dK_{21}}{1-K_{22}}\right)\int_0^1 e^{-\nu x}\left(z - \frac{K_{21}}{1-K_{22}}y(1)\right)y_x(1)dx.$$
$$\leqslant -\begin{pmatrix} \|y\|_{H^2} \\ \|z - \frac{K_{21}}{1-K_{22}}y(1)\|_{H^2}\end{pmatrix}^T M_4 \begin{pmatrix} \|y\|_{H^2} \\ \|z - \frac{K_{21}}{1-K_{22}}y(1)\|_{H^2}\end{pmatrix},$$

with

$$M_4 = \begin{pmatrix} (1-d)\mu e^{-\mu} & -|\frac{\sqrt{2}dK_{21}}{1-K_{22}}| \\ -|\frac{\sqrt{2}dK_{21}}{1-K_{22}}| & \frac{d\nu}{\varepsilon}e^{-\nu}\end{pmatrix}.$$

We note that $M_4 > 0$ for $0 < \varepsilon \leq \varepsilon_2$ for a suitable ε_2. With $\varepsilon^\star = \min\{\varepsilon_1, \varepsilon_2\}$ we got the result. ∎

Let us state the Tikhonov theorem of linear hyperbolic systems

Theorem [Tang, CP, Girard; 2015]

If $\rho(K) < 1$, then $\exists$ positive values $\varepsilon^\star$, C, C', ω $\forall y^0 \in H^2(0,1)$ satisfying the compatibility conditions $y^0(0) = K_r y^0(1)$, $\Lambda_1 y^0_x(0) = K_r \Lambda_1 y^0_x(1)$, and $z^0 = (I - K_{22})^{-1} K_{21} \bar{y}_0(1)$, such that $\forall 0 < \varepsilon < \varepsilon^\star$ and $\forall t \geqslant 0$,

$$\|y(.,t) - \bar{y}(.,t)\|^2_{L^2} \leq C\varepsilon e^{-\omega t} \|\bar{y}_0\|^2_{H^2(0,1)} \tag{16}$$

$$\|z(.,t) - (I_m - K_{22})^{-1} K_{21} \bar{y}(1,t)\|^2_{L^2} \leq C' \varepsilon e^{-\omega t} \|\bar{y}_0\|^2_{H^2(0,1)} \tag{17}$$

Inequality (16) is an approximation of the slow dynamics
Inequality (17) is an approximation of the fast dynamics
Under the assumption $\rho(K) < 1$, all systems are exp. stable.

Sketch of Proof

Let $\eta = y - \bar{y}$, and $\delta = z - (I_m - K_{22})^{-1} K_{21} \bar{y}(1, .)$. Computing the difference of the full system with the reduced and boundary layer systems it holds

$$\begin{gathered}
\eta_t + \Lambda_1 \eta_x = 0 \\
\varepsilon \delta_t + \Lambda_2 \delta_x = \varepsilon (I_m - K_{22})^{-1} K_{21} \Lambda_1 \bar{y}_x(1, .) \\
\begin{pmatrix} \eta(0, t) \\ \delta(0, t) \end{pmatrix} = K \begin{pmatrix} \eta(1, t) \\ \delta(1, t) \end{pmatrix}
\end{gathered}$$

We are going to bound the source term, and to deduce some properties on η and δ.

By trace inequality, $\forall t \geq 0$,

$$\|\bar{y}_x(1,t)\| \leq \sqrt{2}\|\bar{y}(.,t)\|_{H^2(0,1)}$$

and since $\rho(K) < 1$, we have $\rho(K_r) < 1$ and thus, there exist C_r and α such that

$$\|\bar{y}(.,t)^2\|_{H^2(0,1)} \leq C_r e^{-\alpha t}\|\bar{y}_0\|^2_{H^2(0,1)}$$

Let us consider the function $V(\eta,\delta) = \int_0^1 e^{-\mu x}(\eta^\top Q\eta + \varepsilon\delta^\top P\delta)dx$. Selecting P and Q in a suitable way, we get

$$\begin{aligned} \dot{V} &\leq -\gamma V + \varepsilon\beta\|\bar{y}_x(1,t)\|^2 \\ &\leq -\gamma V + \varepsilon\beta C_r e^{-\alpha t}\|\bar{y}_0\|^2_{H^2(0,1)} \end{aligned}$$

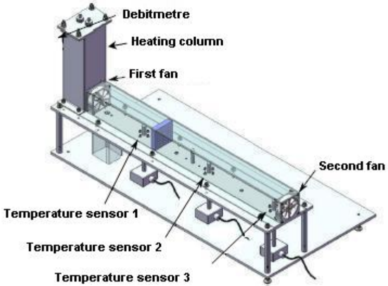
And then use the comparison principle and $\eta(t=0)=0$. ■

Coupled dynamics: fast PDE with ODE:

$$\begin{cases} \dot{y}(t) = Ay(t) + Bz(1,t) \\ \varepsilon z_t + \Lambda z = 0 \\ z(0,t) = K_1 z(1,t) + K_2 y(t), \end{cases}$$

with $y(t) \in \mathbb{R}^n$ and $z(x,t) \in \mathbb{R}^m$, $\varepsilon > 0$ small, A, B... are matrices

Potential application:



The reduced system is

$$\dot{\bar{y}}(t) = (A + BK_r)\bar{y}(t)$$

with $K_r = (I_m)K_1)^{-1}K_2$

The boundary layer system is

$$\bar{z}_t(x,\tau) + \Lambda z(x,\tau) = 0$$
$$z(0,\tau) = K_1 z(1,\tau)$$

with $\tau = t/\varepsilon$.

Assumption #1

The boundary-layer system is so that all eigenvalues of $A + BK_r$ are in the (open) left-part plane.

Assumption #2

The reduced system is so that $\rho(K_1) < 1$.

Theorem

Under Assumptions #1 and #2, the full system is exp. stable in L^2 norm for $\varepsilon > 0$ sufficiently small

Nice case!

Proof: $V(y,z) = y^\top P y + \int_0^1 e^{-\mu x}(z - K_r y)^\top Q (z - K_r y) dx$ where P is a pos. definite matrix and Q is a diagonal pos. definite matrix.

PDE with fast ODE?

What happens with fast dynamics in the boundary conditions?
Can we approximate the fast boundary condition by a static law?
Consider a hyperbolic PDE coupled with a fast ODE:

$$\begin{cases} \varepsilon \dot{z} = Az + By(1) \\ y_t + \Lambda y_x = 0 \\ y(0,t) = K_1 y(1,t) + K_2 z(t), \\ z(0) = z_0 \\ y(x,0) = y_0(x), \end{cases}$$

with $y(x,t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^m$, $\varepsilon > 0$ is small, A, B... are matrices

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with $y(x,t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^m$, $\varepsilon > 0$ is small, A, B... are matrices

The reduced system is

$$\begin{cases} \bar{y}_t(x,t) + \Lambda \bar{y}_x(x,t) = 0 \\ \bar{y}(0,t) = K_r \bar{y}(1,t) \\ \bar{y}(x,0) = y_0(x) \end{cases}$$

with $K_r = K_1 - K_2 A^{-1} B$.
The boundary layer system is

$$\begin{cases} \frac{d\bar{z}(\tau)}{d\tau} = A\bar{z}(\tau) \\ \bar{z}(0) = z_0 + A^{-1} B y_0(1) \end{cases}$$

with $\bar{z} = z + A^{-1} B y(1)$.

Stability analysis for sub-systems

Assumption #1

The reduced system is so that $\rho(K_r) < 1$.

Assumption #2

The boundary-layer system is so that all eigenvalues of A are in the (open) left-part plane.

Conjecture

Assumptions #1 and #2 $\not\Longrightarrow$ the exp. stability of the full dynamics

As in the PDE-PDE case!

To do that consider

$$\begin{cases} \varepsilon \dot{z}(t) = -0.1z(t) - y(1) \\ y_t(x,t) + y_x(x,t) = 0 \\ y(0,t) = 2y(1,t) + 0.2z(t) \end{cases} \tag{18}$$

Assumptions #1 and #2 hold.
The reduced system and the boundary layer system are both exp. stable.
But the full dynamics seems to be unstable
(there exists a solution which diverges on numerical simulations)

Proof of the unstability for (18)?

Assumption #3

$\exists$ P symmetric definite positive matrix, Q diagonal definite positive and $\mu > 0$ such that

$$Q\Lambda - K_r Q \Lambda K_r > 0$$
$$\begin{pmatrix} e^{-\mu} Q\Lambda - K_1^\top Q\Lambda K_1 & -K_1^\top Q\Lambda K_2 - B^\top P \\ -K_2^\top Q\Lambda K_1 - PB & -(A^\top P + PA) - K_2^\top Q\Lambda K_2 \end{pmatrix}$$

Assumption #3 implies

- Assumption #1 on reduced system
- Assumption #2 on boundary layer system
- $\rho(K_1) < 1$ on a the y-component of the full system

Sufficient stability condition and Tikhonov theorem

Theorem

Under Assumption #3, the full system is exp. stable in L^2 norm for $\varepsilon > 0$ sufficiently small

Theorem

Under Assumption #3, $\exists\ C$, $\omega\ \varepsilon^\star$ such that $\forall 0 < \varepsilon < \varepsilon^\star$, $\forall y_0$ in $H^2(0,1)$ satisfying the compatibility condition and for all $z_0 \in \mathbb{R}^m$, it holds, $\forall t \geq 0$,

$$\|y(.,t) - \bar{y}(.,t)\|^2_{L^2(0,1)} \leq \varepsilon C e^{-\omega t}(\|\bar{y}_0\|^2_{H^2(0,1)} + |z_0 + A^{-1}B\bar{y}_0|^2)$$

Main lines of proof:

- consider the error system
- see $\bar{y}_x(1,t)$ as a perturbation
- use H^2 Lyapunov function

The Saint-Venant–Exner system may be rewritten as

$$\begin{cases} \varepsilon W_{1\tilde{t}} + \frac{\lambda_1}{\lambda_2} W_{1x} = 0 \\ W_{2\tilde{t}} + W_{2x} = 0 \\ \varepsilon W_{3\tilde{t}} + W_{3x} = 0 \end{cases} \quad (19)$$

with $\tilde{t} = \lambda_2 t$ and $\varepsilon = \lambda_2/\lambda_3$.
The boundary conditions are

$$\begin{pmatrix} W_1(0,\tilde{t}) \\ W_2(0,\tilde{t}) \\ W_3(0,\tilde{t}) \end{pmatrix} = \begin{pmatrix} 0 & k_{12} & k_{13} \\ k_{21} & 0 & 0 \\ \xi(k_{21}) & 0 & 0 \end{pmatrix} \begin{pmatrix} W_1(1,\tilde{t}) \\ W_2(1,\tilde{t}) \\ W_3(1,\tilde{t}) \end{pmatrix}$$

for $\xi(k_{21}) = -\frac{[(\lambda_1 - V^\star)^2 - gH^\star] + k_{21}[(\lambda_2 - V^\star)^2 - gH^\star]}{(\lambda_3 - V^\star)^2 - gH^\star}$

The reduced system is

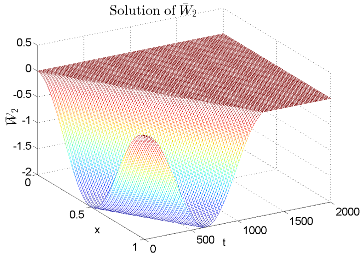
$$\begin{cases} \overline{W}_{2\tilde{t}} + \overline{W}_{2x} = 0 \\ \overline{W}_2(0,\tilde{t}) = K_r \overline{W}_2(1,\tilde{t}) \end{cases} \tag{20}$$

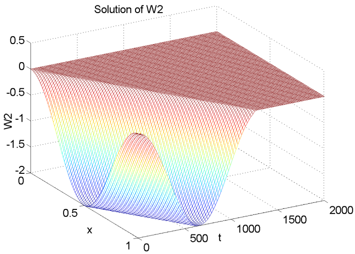
with $K_r = \frac{k_{12}k_{21}}{1-k_{13}\xi(k_{21})}$. We were able to find control gains k_i such that

- $\rho(K) < 1$ and thus the full system is exp. stable
- $K_r = 0$ and thus the slow dynamics converge to the equilibrium in finite-time

This makes the full system converging as fast as we can.

$\varepsilon = 6 \times 10^{-6}$. Numerical scheme may be quite difficult but we know that we could use the subystems





$\overline{W}_2$ of (20) with $K_r = 0$ W_2 of (19) with same control

- Both graphs are roughly the same.
- The finite time of convergence estimated is $T = \frac{1}{\lambda_2}$ which is close to the numerically computed finite time.

Conclusion

- Sufficient stability condition and Tikhonov theorem for linear hyperbolic systems (PDE-PDE and ODE-PDE)
- Boundary control synthesis of a class of linear hyperbolic systems based on the singular perturbation method.
- Slow dynamics has been stabilized in finite time.
- Boundary control design has been achieved for a linearized Saint-Venant–Exner system.

Future works

- Extend this work to systems of balance laws.
- Consider other PDEs:
  quasilinear hyperbolic system, or parabolic equations?

Thank you for your attention

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