Stability for a Transmission Problem in Thermoviscoelasticity

Reinhard Racke

University of Konstanz

Conference on Stability of Non-Conservative Systems
Valenciennes, July 4-7, 2016

Joint work with: Jaime E. Muñoz Rivera

Setting

$$\Omega_2 \subset \Omega_1 \subset \mathbb{R}^d$$

Ω_2: purely elastic material. Ω_1: thermoviscoelastic material.

$$\begin{aligned} u_{tt} + E_1 u + \beta E_1 u_t + \nabla\theta &= 0 \quad \text{in} \quad \Omega_1 \times (0,\infty), && (1) \\ \theta_t - \Delta\theta + \operatorname{div} u_t &= 0 \quad \text{in} \quad \Omega_1 \times (0,\infty), && (2) \\ \rho_2 v_{tt} + E_2 u &= 0 \quad \text{in} \quad \Omega_2 \times (0,\infty), && (3) \end{aligned}$$

where

$$E_j := -\mu_j \Delta - (\mu_j + \delta_j)\nabla \operatorname{div}, \quad j = 1, 2.$$

Setting

Transmission conditions on the interface $\Gamma_1 = \partial\Omega_2$:

$$u = v \quad \text{on} \quad \Gamma_1 \times (0,\infty), \tag{4}$$

$$\partial_\nu^{E_1} u + \theta\,\nu + \beta\partial_\nu^{E_1} u_t = \partial_\nu^{E_2} v \quad \text{on} \quad \Gamma_1 \times (0,\infty), \tag{5}$$

where

$$\partial_\nu^{E_j} = -\mu_j \partial_\nu - (\mu_j + \delta_j)\nu\, \mathrm{div}\,,$$

and

$$\partial_\nu = \nu\nabla.$$

Setting

Additional boundary conditions on Γ_1 and on $\Gamma_0 = \partial\Omega_1 \setminus \Gamma_1$:

$$u = 0, \quad \theta = 0 \quad \text{on} \quad \Gamma_0 \times (0, \infty), \tag{6}$$

$$\partial_\nu \theta = 0 \quad \text{on} \quad \Gamma_1 \times (0, \infty). \tag{7}$$

Initial conditions,

$$u(\cdot, 0) = u^0, \quad u_t(\cdot, 0) = u^1, \quad \theta(\cdot, 0) = \theta^0 \quad \text{in} \quad \Omega_1, \tag{8}$$

$$v(\cdot, 0) = v^0, \quad v_t(\cdot, 0) = v^1 \quad \text{in} \quad \Omega_2. \tag{9}$$

Comparison

Results hold if the elastic operator E_1 is replaced by the scalar Laplace operator.

Exponential stability is given if

- The Kelvin-Voigt damping $\beta\Delta u_t$ is replaced by βu_t, or
- the wave equation $u_{tt} - u_{xx} - \beta u_{xxt}$ is replaced by $u_{tt} + u_{xxxx} + u_{xxxxt}$, i.e., E_1 is replaced by ∂_x^4.

Applications

Applications are given for the optimal design of materials in creating damping effects in

– noise reduction in cars and airplanes,

– bridges.

Well-posedness

$$W := (u, v, u_t, v_t, \theta)'$$

leads to a semigroup

$$S(t) = e^{\mathcal{A}t}, \qquad t \geq 0,$$

on the Hilbert space

$$\begin{aligned} \mathcal{H} &= \{ (u,v) \in \left(H^1(\Omega_1)\right)^d \times \left(H^1(\Omega_2)\right)^d \mid u = 0 \text{ on } \Gamma_0, \\ &\quad u = v \text{ on } \Gamma_1 \} \times \left(L^2(\Omega_1)\right)^d \times \left(L^2(\Omega_2)\right)^d \times L^2(\Omega_1). \end{aligned}$$

$\mathcal{A}$ is dissipative.

Theorem

Theorem

$\left(e^{\mathcal{A}t}\right)_{t\geq 0}$ *is not exponentially stable.*

Idea of the proof:

- Consider a related semigroup

$$S_0(t) = e^{\mathcal{A}_0 t} \qquad \text{with essential radius } \; r_{ess}(S_0) \geq 1.$$

- Show for $t_0 > 0$:

$$S(t_0) - S_0(t_0) \quad \text{is compact.}$$

- Then

$$r_{ess}(S) = r_{ess}(S_0) \geq 1.$$

- Then $S(t) \geq 1$.

Weyl's theorem

Background: Weyl's theorem on compact perturbations of essential spectra:

$$B_1 - B_2 \text{ compact } \Longrightarrow \sigma_{ess}(B_1) = \sigma_{ess}(B_2).$$

General case

$$\emptyset \not\subseteq \mathcal{H}_2 \subset \mathcal{H}_1$$

Hilbert spaces.

$$P : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$$

orthogonal projection.

$$S_j = (S_j(t))_{t \geq 0}$$

C_0-semigroup on $\mathcal{H}_j$, for $j = 1, 2$.

Assumption CP: *There is $t_0 > 0$ such that for $t \geq t_0$ we have*
(i) $r_{ess}(S_2(t)) \geq 1$,
(ii) $S_1(t) - S_2(t) : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ is compact.

General case

Theorem

If Assumption CP holds, S_1 is not exponentially stable.

PROOF:

$$\begin{aligned} 1 &\leq r_{ess}(S_2(t)P) = r_{ess}(S_1(t)P) \leq \|S_1(t)P\|_{\mathcal{H}_1} \\ &\leq \|S_1(t)\|_{\mathcal{H}_1}. \end{aligned}$$

□

Associated semigroup S_0

$$\begin{aligned} \widetilde{u}_{tt} + E_1\widetilde{u} + \beta E_1\widetilde{u}_t + \nabla\widetilde{\theta} &= 0 \quad \text{in} \quad \Omega_1 \times (0,\infty), \qquad (10) \\ \widetilde{\theta}_t - \Delta\widetilde{\theta} + \operatorname{div}\widetilde{u}_t &= 0 \quad \text{in} \quad \Omega_1 \times (0,\infty), \qquad (11) \end{aligned}$$

with boundary conditions

$$\widetilde{u} = 0, \quad \widetilde{\theta} = 0 \qquad \text{on} \quad (\Gamma_0 \cup \Gamma_1) \times (0,\infty), \qquad (12)$$

as well as

$$\widetilde{v}_{tt} + E_2\widetilde{v} = 0 \quad \text{in} \quad \Omega_2 \times (0,\infty), \qquad (13)$$

with boundary conditions

$$\widetilde{v} = 0 \qquad \text{on} \quad \Gamma_1 \times (0,\infty), \qquad (14)$$

plus initial conditions.

Associated semigroup S_0

This uncoupled system defines S_0 resp. $\mathcal{A}_0$ on the Hilbert space

$$\mathcal{H}_0 := \left(H_0^1(\Omega_1)\right)^d \times \left(H_0^1(\Omega_2)\right)^d \times \left(L^2(\Omega_1)\right)^d \times \left(L^2(\Omega_2)\right)^d \times L^2(\Omega_1).$$

It is energy conserving in the v-variable, hence $r_{ess}(S_0) \geq 1$. Choose

$$\mathcal{H}_1 := \mathcal{H},\ \mathcal{H}_2 := (\{0\})^d \times \left(H_0^1(\Omega_2)\right)^d \times (\{0\})^d \times \left(L^2(\Omega_2)\right)^d \times \{0\}$$

and

$$S_1 := S,\ S_2 := S_{0/\mathcal{H}_2}.$$

Previous theorem applies. Prove compactness!

Compactness

Use

Lemma

$$w_{tt} + E_2 w = f \quad \text{in} \quad \Omega_1 \times (0,T),$$

and

$$w = 0 \quad \text{on} \quad \Gamma_1 \times (0,T).$$

implies

$$\int_0^T \int_{\Gamma_1} |\partial_\nu w|^2 + |\operatorname{div} w|^2 \, do \, ds \leq$$
$$C_T \left(\int_0^T \int_{\Omega_2} |w_t|^2 + |\nabla w|^2 + |f|^2 \, dx \, ds + \int_{\Omega_2} |w_t(\cdot,0)|^2 + |\nabla w(\cdot,0)|^2 \, dx \right).$$

Strong stability

Theorem

$$\forall\ \Phi^0 \in \mathcal{H} :\ e^{\mathcal{A}t}\phi^0 \to 0 \qquad as\ t \to \infty.$$

PROOF: Show

$$i\mathbb{R} \subset \rho(\mathcal{A})$$

by contradiction. Use: Unique continuation principle for isotropic elasticity.

Polynomial stability

Theorem

The semigroup $\left(\mathrm{e}^{t\mathcal{A}}\right)_{t\geq 0}$ *decays polynomially of order at least* $\frac{1}{3}$, *i.e.*

$$\exists\, C > 0\; \exists\, t_0 > 0\; \forall\, t \geq t_0\; \forall\, \Phi^0 \in D(\mathcal{A}) :$$

$$\|\mathrm{e}^{t\mathcal{A}}\Phi_0\|_{\mathcal{H}} \leq C\, t^{-\frac{1}{3}}\, \|\mathcal{A}\Phi^0\|_{\mathcal{H}}.$$

Steps of the proof

Lemma

The following characterizations (15) and (16), for a semigroup in a Hilbert space $\mathcal{H}_1$ with generator $\mathcal{A}_1$, are equivalent, where the parameters $\alpha, \beta \geq 0$ are fixed:

$$\exists\, C > 0\ \exists\, \lambda_0 > 0\ \forall\, \lambda \in \mathrm{Re},\ |\lambda| \geq \lambda_0\ \forall\, F \in D(\mathcal{A}_1^\alpha) :$$

$$\| \left(i\lambda - \mathcal{A}_1\right)^{-1} F\|_{\mathcal{H}} \leq C|\lambda|^\beta \|\mathcal{A}_1^\alpha F\|_{\mathcal{H}}, \tag{15}$$

$$\exists\, C > 0\ \exists\, t_0 > 0\ \forall\, t \geq t_0\ \forall\, \Phi^0 \in D(\mathcal{A}_1) :$$

$$\|\mathrm{e}^{t\mathcal{A}_1}\Phi_0\|_{\mathcal{H}} \leq C\, t^{-\frac{1}{\alpha+\beta}} \|\mathcal{A}_1\Phi^0\|_{\mathcal{H}}. \tag{16}$$

It is shown that (15) holds with $\alpha = 1,\ \beta = 2$.

Steps of the proof

Let $(i\lambda - \mathcal{A})\,\Phi = F = (f_1, f_2, f_3, f_4, f_5)' \in D(\mathcal{A})$. Denoting $\Phi = (u, v, U, V, \theta)'$:

$$
\begin{aligned}
i\lambda u - U &= f_1, && (17)\\
i\lambda v - V &= f_2, && (18)\\
i\lambda U + E_1 u + \beta E_1 U + \nabla\theta &= f_3, && (19)\\
i\lambda V + E_2 v &= f_4, && (20)\\
i\lambda\theta - \Delta\theta + \operatorname{div} U &= f_5. && (21)
\end{aligned}
$$

Steps of the proof

Estimates for u, U, θ are easy because of the dissipativity. For v, V: One needs to estimate normal and tangential derivatives on Γ_0.

It appears from the transmission conditions:

$$\|f_2\|_{H^{\frac{3}{2}}(\Gamma_1)} \leq C\|f_2\|_{H^2(\Omega_2)} \leq C\|\mathcal{A}F\|_{\mathcal{H}}. \qquad (\alpha = 1)$$

Powers of $|\lambda|$ $(\beta = 2)$ arise through the differential equations.

Remarks

- Results hold if the temperature is neglected or the equations of elasticity are replaced by the wave equation.
- The inverse of $\mathcal{A}$ is expected to be non-compact.