Spectral analysis of the Schrödinger operator on binary tree-shaped networks and applications.

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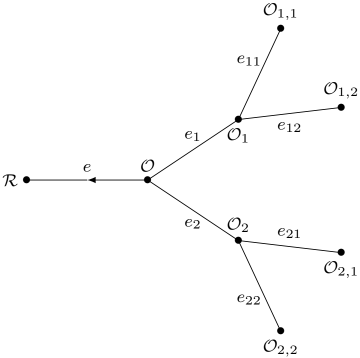
- Motivation: stabilization i.e. can the solution of our evolution problem be guided to a desired final configuration, asymptotically in time?
- If possible, does the energy decrease to zero or to a positive value? Polynomially? Exponentially? The decay rate depends on the spectrum.

Structure of the talk

1. The system: abstract setting and well-posedness
   1. The tree-shaped network
   2. The problem
   3. Comparison with the finite dimensional case
2. Spectral analysis
   1. The conservative and dissipative operators
   2. The iterative approach to get the spectrum
   3. The eigenvalues with their multiplicity
3. Energy decreasing (using a Riesz basis)

A binary tree-shaped network



$$\frac{\partial u_{\bar\alpha}}{\partial t}(x,t) + i\,\frac{\partial^2 u_{\bar\alpha}}{\partial x^2}(x,t) = 0, \quad 0 < x < 1,\ t > 0,\ \bar\alpha \in I, \tag{1}$$

$$i\,u(1,t) + \frac{\partial u}{\partial x}(1,t) = 0,\ u_{\bar\alpha}(0,t) = 0,\ \bar\alpha \in I_{Dir},\ t > 0, \tag{2}$$

$$u_{\bar\alpha\circ(\beta)}(1,t) = u_{\bar\alpha}(0,t), \quad t > 0,\ \beta = 1,2,\ \bar\alpha \in I_{Int}, \tag{3}$$

$$\sum_{\beta=1}^{2} \frac{\partial u_{\bar\alpha\circ(\beta)}}{\partial x}(1,t) = \frac{\partial u_{\bar\alpha}}{\partial x}(0,t), \quad t > 0,\ \bar\alpha \in I_{Int}, \tag{4}$$

$$u_{\bar\alpha}(x,0) = \left(u_{\bar\alpha}\right)_0(x), \quad 0 < x < 1,\ \bar\alpha \in I, \tag{5}$$

where $u_{\bar\alpha} : [0,1] \times (0,+\infty) \to IR$, $\bar\alpha \in I$, is the transverse displacement of the edge $e_{\bar\alpha}$.

In our example, $I_{Dir} := \{(1,1),(1,2),(2,1),(2,2)\}$, $I_{Int} := \{\emptyset,(1),(2)\}$ and $I := I_{Dir} \cup I_{Int}$.

The space $H := \prod_{\bar{\alpha}\in I} L^2(0,1)$ is equipped with the inner product

$$< \underline{u}, \underline{\tilde{u}} >_H := \sum_{\bar{\alpha}\in I} \int_0^1 u_{\bar{\alpha}}(x)\, \bar{\tilde{u}}_{\bar{\alpha}}(x)\, dx. \tag{6}$$

The previous system is reformulated as the first order evolution equation:

$$\underline{u}'(t) = \mathcal{A}_d \underline{u}(t), \text{with } \underline{u}(0) = \underline{u}_0, \tag{7}$$

where the operator $\mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \subset H \to H$ is

$$\mathcal{A}_d \underline{u} := (-i\, \partial_x^2 u_{\bar{\alpha}})_{\bar{\alpha}\in I},$$

$$\mathcal{D}(\mathcal{A}_d) := \left\{ \underline{u} \in \prod_{\bar{\alpha}\in I} H^2(0,1) \ : \underline{u} \text{ satisfies (8), (9) and (10) hereafter} \right\}.$$

$$i\,u(1)+\frac{du}{dx}(1)=0,\ u_{\bar\alpha}(0)=0,\ \bar\alpha\in I_{Dir},\tag{8}$$

$$u_{\bar\alpha\circ(\beta)}(1)=u_{\bar\alpha}(0),\ \beta=1,2,\tag{9}$$

$$\sum_{\beta=1}^{2}\frac{du_{\bar\alpha\circ(\beta)}}{dx}(1)=\frac{du_{\bar\alpha}}{dx}(0),\qquad \bar\alpha\in I_{Int}.\tag{10}$$

The natural energy $E(t)$ of a solution $\underline{u}=(u_{\bar\alpha})_{\bar\alpha\in I}$ is:

$$E(t):=\frac{1}{2}\sum_{\bar\alpha\in I}\int_0^1 |u_{\bar\alpha}(x,t)|^2\,dx.\tag{11}$$

It is proved to be a non-increasing function of the variable t.

(i) For an initial datum $\underline{u}_0 \in H$, there exists a unique solution $\underline{u} \in C([0, +\infty), H)$ to the latter problem. Moreover, if $\underline{u}_0 \in \mathcal{D}(\mathcal{A}_d)$, then

$$\underline{u} \in C([0, +\infty), \mathcal{D}(\mathcal{A}_d)) \cap C^1([0, +\infty), H).$$

(ii) The solution $\underline{u}$ with initial datum in $\mathcal{D}(\mathcal{A}_d)$ satisfies the dissipation law:

$$E'(t) = -\left|u(1,t)\right|^2 \leq 0, \tag{12}$$

If A is a square matrix of order N on $\mathbb{C}$ and if A is skew-Hermitian (i.e. $A^* = -A$), the eigenvalues λ_i are all purely imaginary and their geometric multiplicity is equal to the algebraic multiplicity. The matrix A is unitarily similar to a diagonal matrix.

The solution of $x'(t) = Ax(t)$, with $x(0) = x^0$, is:

$$x(t) = e^{At}x^0 = \sum_{1 \leq i \leq N} e^{\lambda_i \cdot t} x_i^0 \phi_i, \ \forall\ t > 0 \tag{13}$$

where $x^0 = \sum_{1 \leq i \leq N} x_i^0 \phi_i$ is the decomposition of x^0 in the orthonormal basis $(\phi_i)_i$ of the eigenfunctions.

The solution satisfies: $\forall\ t > 0$,

$$\|x(t)\|^2 = \sum_{1 \leq i \leq N} \left| e^{\lambda_i \cdot t} \right|^2 \cdot |x_i^0|^2 \cdot \|\phi_i\|^2 = \|x^0\|^2, \tag{14}$$

More generally, the minimal polynomial of A is: $\pi_A(X) = \prod_{i=1}^{r}(X - \lambda_i)^{s_i}$. The matrix A is similar to a block diagonal matrix where each block is a Jordan block.
Thus the solution of the previous first order evolution equation becomes

$$x(t) = e^{At}x^0 = \sum_{\substack{1\leq i\leq r \\ 0\leq k\leq |J_{\lambda_i}|-1}} e^{\lambda_i\cdot t}t^k v_{i,k},\ \forall\ t > 0, \tag{15}$$

where $v_{i,k}$ belongs to $Ker[(A-\lambda_i)^{s_i}]$ (characteristic space).
To get the estimate for $\|x(t)\|^2$, we need to know more about the algebraic and geometric multiplicity of each eigenvalue.

Let us come back to our problem. The conservative operator, associated to the dispersive operator $\mathcal{A}_d$ is called $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subset H \to H$. It is:

$$\mathcal{A}_0 \underline{u} := (-i\, \partial_x^2 u_{\bar{\alpha}})_{\bar{\alpha} \in I},$$

$$\mathcal{D}(\mathcal{A}_0) := \left\{ \underline{u} \in \prod_{\bar{\alpha} \in I} H^2(0,1) \ : \underline{u} \text{ satisfies } (16), (9) \text{ and } (10) \right\}.$$

where the following condition (16) replaces the dissipative condition (8):

$$\frac{du}{dx}(1) = 0,\ u_{\bar{\alpha}}(0) = 0,\ \bar{\alpha} \in I_{Dir}. \tag{16}$$

The operator $\mathcal{A}_0$ is skew-adjoint. Thus the energy of the solution of the first order evolution equation $\underline{u}'(t) = \mathcal{A}_0 \underline{u}$, with $\underline{u}(0) = \underline{u}_0$ satisfies $E(t) = E(0),\ \forall\ t > 0$.

- Necessity to localize the spectrum: since $\mathcal{A}_d$ is dissipative, its eigenvalues have a negative real part. But they are not in finite number and can tend to the imaginary axis.
- Do some eigenvalues lie on the imaginary axis? Can we get information on the multiplicity of the eigenvalues (in both senses: algebraic and geometric)?
- Can we obtain a decomposition similar to (15)?
- Yes, if we view $\mathcal{A}_d$ as a perturbation of $\mathcal{A}_0$ and apply a reformulation of Guo's version of Bari Theorem to prove that some eigenfunctions of the operator $\mathcal{A}_d$ form a Riesz basis of the subspace they span.

- A basis $\{f_n\}$ for a Hilbert space H is a Riesz basis for H if it is equivalent to some (and therefore every) orthonormal basis for H. By "equivalent", we mean there exists a topological isomorphism $S : H \to H$ such that $\{Sf_n\}$ is an orthonormal basis of H.
- In particular, an orthonormal basis of a Hilbert space is a Riesz basis.
- To prove that some eigenfunctions of the operator $\mathcal{A}_d$ form a Riesz basis of the subspace they span, we need the eigenfunctions of $\mathcal{A}_d$ to be quadratically close to those of $\mathcal{A}_0$, except from a finite number of eigenfunctions. It is the assumption of Bari Theorem.
- We use an iterative strategy to get the required information on the spectrum of $\mathcal{A}_d$.

The eigenvalue problem: $\lambda = i\omega^2$ ($\omega \in \mathbb{C}^*$) is an eigenvalue of $\mathcal{A}_d$ with associated eigenvector $\underline{\phi} \in \mathcal{D}(\mathcal{A}_d)$ if and only if $\underline{\phi}$ satisfies the transmission and boundary conditions (8), (9) and (10) and

$$\mathcal{A}_d\underline{\phi} = (-i\,\partial_x^2\phi_{\bar{\alpha}})_{\bar{\alpha}\in I} = i\omega^2(\phi_{\bar{\alpha}})_{\bar{\alpha}\in I} \Leftrightarrow \forall\ \bar{\alpha} \in I, \partial_x^2\phi_{\bar{\alpha}} = -\omega^2\phi_{\bar{\alpha}}.$$

Introducing the vector $F_{\bar{\alpha}}(x) := \begin{pmatrix} \phi_{\bar{\alpha}}(x) \\ \partial_x\phi_{\bar{\alpha}}(x) \end{pmatrix}$ to reduce the order of the eigenvalue problem as well as $M(\omega) := \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$, it becomes:

$$(EP): \quad F'_{\bar{\alpha}}(x) = M(\omega)F_{\bar{\alpha}}(x) \text{ on } (0,1), \ \ \forall\bar{\alpha} \in I.$$

Thus $F_{\bar\alpha}(x) = e^{M(\omega)x} F_{\bar\alpha}(0)$ where $e^{M(\omega)x} = \begin{pmatrix} \cos(\omega x) & \dfrac{\sin(\omega x)}{\omega} \\ -\omega \sin(\omega x) & \cos(\omega x) \end{pmatrix}$.

For simplicity, the computations to establish the characteristic equation are presented in the case $I_{Dir} = \{(1,1); (1,2); (2,1); (2,2)\}$.

The Dirichlet condition at the leaves of the tree implies $F_{\bar\alpha}(0) = \begin{pmatrix} 0 \\ \partial_x \phi_{\bar\alpha}(0) \end{pmatrix}$ for $\bar\alpha \in I_{Dir}$. Then $F_{\bar\alpha}(1) = e^{M(\omega)} F_{\bar\alpha}(0)$ i.e. $F_{\bar\alpha}(1) = \phi'_{\bar\alpha}(0) \begin{pmatrix} \dfrac{\sin(\omega)}{\omega} \\ \cos(\omega) \end{pmatrix}$, for $\bar\alpha \in I_{Dir}$ and $\omega \neq 0$.

Now the continuity at the interior vertices $\mathcal{O}_1$ and $\mathcal{O}_2$ implies:

$$\phi'_{j,1}(0) \cdot \frac{\sin(\omega)}{\omega} = \phi'_{j,2}(0) \cdot \frac{\sin(\omega)}{\omega} \text{ for } j = 1 \text{ and } j = 2. \tag{17}$$

Either $\omega = k\pi$ with $k \in \mathbb{Z}^*$ (first family of eigenvalues) or the following condition is imposed:

$$\phi'_{j,1}(0) = \phi'_{j,2}(0) \text{ for } j = 1 \text{ and } j = 2. \tag{18}$$

Then the second transmission condition at the interior vertices $\mathcal{O}_1$ and $\mathcal{O}_2$ (condition (10)) implies, for $j = 1$ and $j = 2$:

$$F_j(0) = \phi'_{j,1}(0) \begin{pmatrix} \dfrac{\sin(\omega)}{\omega} \\ 2\cos(\omega) \end{pmatrix} \text{ and thus } F_j(1) = \phi'_{j,1}(0) \begin{pmatrix} 3\cos(\omega)\dfrac{\sin(\omega)}{\omega} \\ -\sin^2(\omega) + 2\cos^2(\omega) \end{pmatrix}.$$

It follows from the continuity at the vertex $\mathcal{O}$: either $\omega = \dfrac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$ (second family of eigenvalues) or

$$\phi'_{1,1}(0) = \phi'_{2,1}(0). \tag{19}$$

Either $\omega = k\pi, k \in \mathbb{Z}^*$, or $\omega = (\pi/2) + k\pi, k \in \mathbb{Z}$, or (18) and (19) are imposed. Thus

$$\phi'_{1,1}(0) = \phi'_{1,2}(0) = \phi'_{2,1}(0) = \phi'_{2,2}(0) =: v. \quad (20)$$

The "eigenvector" $F_{\bar{\alpha}}$ for $\bar{\alpha}$ in I_{Dir} is: $F_{\bar{\alpha}}(x) := \begin{pmatrix} 0 \\ \partial_x \phi_{\bar{\alpha}}(0) \end{pmatrix} = v \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Thus the dimension of the corresponding eigenspace is one i.e. the geometric multiplicity of these eigenvalues is one.

- The geometric multiplicity of $k\pi$ is $2^n - 1$ for both $\mathcal{A}_0$ and $\mathcal{A}_d$ (3 pour $n = 2$).
- The geometric multiplicity of $(\pi/2) + k\pi$ is equal to: $\frac{1}{3}2^n + \frac{2}{3}$ if n is even and $\frac{1}{3}(2^n - 2)$ if n is odd for $\mathcal{A}_0$ (2 pour $n = 2$).
  It is $\frac{1}{3}(2^n - 1)$ if n is even and $\frac{1}{3}(2^n - 2)$ if n is odd for $\mathcal{A}_d$ (1 pour $n = 2$).

We know that $F_1(1) = F_2(1)$ (it comes from (19)) and have their expression. Applying the second transmission condition at the vertex $\mathcal{O}$, leads to multiplying the second component of $F_1(1)$ by the number 2. Then $F(1) = e^{M(\omega)}F(0)$. It is:

$$F(1) := v \begin{pmatrix} \frac{\sin(\omega)}{\omega}\left(7\frac{\cos^2(\omega)}{\omega} - 2\sin^2(\omega)\right) \\ \cos(\omega)\left(4\cos^2(\omega) - 5\sin^2(\omega)\right) \end{pmatrix}.$$

The condition at the root of the tree leads to:

$$i\epsilon\frac{\sin(\omega)}{\omega}\left(7\frac{\cos^2(\omega)}{\omega} - 2\sin^2(\omega)\right) + \cos(\omega)\left(4\cos^2(\omega) - 5\sin^2(\omega)\right) = 0 \quad (21)$$

where $\epsilon = 0$ for the conservative operator and $\epsilon = 1$ for the dissipative operator.

How to determine the simple eigenvalues of $\mathcal{A}_0$ and $\mathcal{A}_d$? (2)

Using Euler's formula with $z := e^{2i\omega}$, the characteristic equation (21) with $\epsilon = 0$ is:

$$(z+1)(4(z+1)^2 + 5(z-1)^2) = 0 \Leftrightarrow 9z^3 + 7z^2 + 7z + 9 = 0. \qquad (22)$$

The roots of this polynomial give the (algebraically) simple eigenvalues of the conservative operator $\mathcal{A}_0$.

The eigenvalues in the case $n = 2$

- For both the conservative and dissipative operators, the purely imaginary eigenvalues: $i\omega^2 = ik^2\pi^2, k \in \mathbb{Z}^*$ and $i\omega^2 = i[(\pi/2) + k\pi]^2, k \in \mathbb{Z}$.
- For the conservative operator $\mathcal{A}_0$, the values $\lambda = i\omega^2$, where $z := e^{2i\omega}$ satisfies $P_{A,2}(z) = 0$, with $P_{A,2}(z) := 9z^3 + 7z^2 + 7z + 9$.
- For the dissipative operator $\mathcal{A}_d$, the values $\lambda = i\omega^2$, where $z := e^{2i\omega}$ satisfies $P_{A,2}(z) + \frac{1}{\omega} P_{B,2}(z) = 0$, with $P_{A,2}(z) := 9z^3 + 7z^2 + 7z + 9$ and $P_{B,2}(z) := 9z^3 + z^2 - z - 9$.

Let σ be the spectrum of the dissipative operator $\mathcal{A}_d$, then $\sigma = \sigma_1 \cup \sigma_2 \cup \tilde{\sigma_2}$, where

$$\begin{aligned}
\sigma_1 &= \{i(k\pi)^2 : k \in \mathbb{Z}^*\} \cup \left\{ i \left(k\pi + \frac{\pi}{2}\right)^2 : k \in \mathbb{Z} \right\} \\
\tilde{\sigma_2} &= \{(\lambda_k)_{k \in S} : S \text{ is finite, } \Re(\lambda_k) < 0\} \\
\sigma_2 &= \{i(\omega_{j,k})^2 : j = 1, 2, 3, k \in \mathbb{Z}, |k| \geq k_0\}, k_0 \text{ being an integer.}
\end{aligned}$$

Moreover

$$\Re(i(\omega_{j,k})^2) < 0, \; \forall i(\omega_{j,k})^2 \in \sigma_2$$

and the following asymptotic behaviour holds:

$$i(\omega_{j,k})^2 = i \left(k^2\pi^2 + k\pi \arg(z_{A,j}^{(2)}) + \frac{(\arg(z_{A,j}^{(2)}))^2}{4} \right) + 2\pi\gamma_j + o(1) \qquad (23)$$

Localization of the eigenvalues of the dispersive operator in the case $n = 2$ (2)

where γ_j is the real negative number defined by: $\gamma_j = -\dfrac{P_{B,2}(z_{A,j}^{(2)})}{2\pi z_{A,j}^{(2)} (P_{A,2})'(z_{A,j}^{(2)})}$.

The polynomial $P_{A,2}$ admits 3 distinct complex roots $z_{A,j}^{(2)} \neq 1,\ j = 1, 2, 3$ with modulus equal to 1. They are:

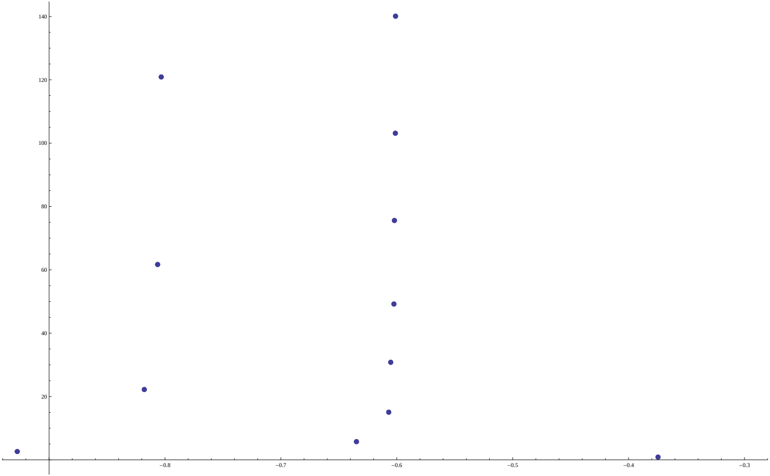
$$z_{A,1}^{(2)} = -1 = e^{i\pi},\ z_{A,2}^{(2)} = \frac{1}{9}(1 - 4i\sqrt{5}) = e^{-i\arctan(4\sqrt{5})},\ z_{A,3}^{(2)} = e^{i\arctan(4\sqrt{5})}.$$

Then the set σ_2 has two vertical asymptots:

$$\Re(\lambda) = 2\pi\gamma_1 = -\frac{4}{5},\ \Re(\lambda) = 2\pi\gamma_2 = 2\pi\gamma_3 = -\frac{3}{5}.$$

At last, numerically the eigenvalue of $\mathcal{A}_d$ with the largest real part is $\lambda \approx -0.37459 + 0.873125i$.

The spectrum for $n = 2$



Theorem

Let $E(t)$ and H be defined as in the introduction. Let H_1 (respectively H_2) be the subspace of H spanned by the $\underline{\psi}^1(\omega,\cdot)$'s (resp. $\underline{\psi}^2(\omega,\cdot)$'s), which are the normalized (in H) eigenfunctions of $\mathcal{A}_d$ associated to the eigenvalues $i\omega^2$ in σ_1 (resp. $\sigma_2 \cup \tilde{\sigma_2}$).

1. *H_1 is orthogonal to H_2.*
2. *Let $\underline{u}_0$ in H and $\underline{u}_0^1$ its orthogonal projection onto H_1. Then $E(t)$ decreases to $E_1(0) := \|\underline{u}_0^1\|_H^2$ when t tends to $+\infty$. More precisely there exists a constant $C > 0$ such that*
   $E(t) \leq E_1(0) + Ce^{-2\beta t}E_2(0)$

where $-\beta := \sup_{\{i\omega^2 \in (\sigma_2 \cup \tilde{\sigma_2})\}} \Re(i\omega^2) < 0$.

After it has been proved that the $\underline{\psi}^2(\omega,\cdot)$'s form a Riesz basis of H_2 and since the $\underline{\psi}^1(\omega,\cdot)$'s form an orthonormal basis of H_1, the initial condition $\underline{u}_0 := \left((u_{\bar{\alpha}})_{\bar{\alpha}\in I}\right)_0$ is written as a sum of two terms:

$$\underline{u}_0 := \underline{u}_0^1 + \underline{u}_0^2 = \sum_{i\omega^2\in\sigma_1} \underline{u}_0^1(\omega,\cdot)\underline{\psi}^1(\omega,\cdot) + \sum_{i\omega^2\in(\sigma_2\cup\tilde{\sigma_2})} \underline{u}_0^2(\omega,\cdot)\underline{\psi}^2(\omega,\cdot).$$

The solution of the boundary value problem given in the introduction is:
$u(t) = e^{\mathcal{A}_d t}\underline{u}_0 := \underline{u}^1(t) + \underline{u}^2(t)$.
The energy, defined in the introduction, by (11) is: $E(t) = E_1(t) + E_2(t)$ (since H_1 is orthogonal to H_2), with, for any $t \geq 0$:

$$E_1(t) := \|\underline{u}^1\|_{H_1}^2 = \sum_{i\omega^2\in\sigma_1} \left\|\underline{u}_0^1(\omega,\cdot)\right\|_{H_1}^2 |e^{i\omega^2 t}|^2 = E_1(0)$$

since σ_1 contains only purely imaginary eigenvalues.

The set σ_2 contains the "large" eigenvalues, which have an algebraic multiplicity equal to 1, due to Rouché's Theorem.
The algebraic multiplicity of an eigenvalue in $\tilde{\sigma_2}$ (the finite set of the "small" eigenvalues) is also one. This is a consequence of the proof of Bari's Theorem (as it is given in a paper by Abdallah/Mercier/Nicaise). Now, since the $\underline{\psi}^2(\omega,\cdot)$'s form a Riesz basis of H_2, there exists $C>0$, such that, for any $t \geq 0$:

$$E_2(t) := \|\underline{u}^2(t)\|^2_{H_2} \leq C \sum_{i\omega^2 \in \tilde{\sigma_2} \cup \sigma_2} \left\| \underline{u}_0^2(\omega,\cdot) \right\|^2_{H_2} |e^{i\omega^2 t}|^2.$$

The set σ_2 has at most $(n+1)$ vertical asymptots: $Re(\lambda) = 2\pi\gamma_j$, with $j \in \{1, \ldots, n+1\}$. Define j_0 by $2\pi\gamma_{j_0} := \sup_{\{j\in\{1,\ldots,n+1\}\}}(2\pi\gamma_j) < 0$. Thus, if ω is such that $i\omega^2 \in \sigma_2$:

$$|e^{i\omega^2 t}|^2 \leq e^{2\pi\gamma_{j_0} t}.$$

Since $\tilde{\sigma_2}$ is a finite set, the real part of $i\omega^2$ is bounded from above by $-\alpha < 0$ if $i\omega^2 \in \tilde{\sigma_2}$. Thus, if ω is such that $i\omega^2 \in \tilde{\sigma_2}$:

$$|e^{i\omega^2 t}|^2 \leq e^{-2\alpha t}.$$

Hence the result, where $-\beta$ is the maximum between $2\pi\gamma_{j_0}$ and -2α.
For $n = 2$, $-2\beta = -0.74918$ is the energy decay rate.

- dependence of β on n
- generalization to a non-binary tree-shaped network...

- Multistructures: F. Ali Mehmeti, J. von Below, J. Lagnese, G. Leugering, G. Lumer, S. Nicaise, E.J.P.G. Schmidt...
- Stabilization on networks: K. Ammari, C. Castro, A. Haraux, M. Jellouli, M. Khenissi, V. Komornik, D. Mercier, M. Mehrenberger, M. Tucsnak, J. Valein, G.Q. Xu, Y. Zhang, E. Zuazua...
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