A spectral inequality for the bi-Laplace operator

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A spectral inequality for Laplace operator

Let Ω be an open set in $\mathbb{R}^d$.
Let Δ be the Laplace operator. (Or a self-adjoint elliptic operator of order 2).
There exist $\phi_j \in H^2(\Omega) \cap H^1_0(\Omega)$ and $\omega_j > 0$, such that

$$\begin{cases} -\Delta \phi_j = \omega_j \phi_j \text{ in } \Omega, \\ \phi_j = 0 \text{ on } \partial\Omega, \end{cases}$$

and $(\phi_j)_j$ is an orthonormal basis of $L^2(\Omega)$.

Theorem (Jerison-Lebeau)

Let $\mathscr{O}$ be an open subset of Ω. There exists $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq Ce^{C\omega^{1/2}} \|u\|_{L^2(\mathscr{O})}, \quad \omega > 0, \quad u \in \text{Span}\{\phi_j;\ \omega_j \leq \omega\}.$$

A spectral inequality for bi-Laplace operator: hinged boundary conditions

Previous basis $(\phi_j)_j$ satisfies

$$\left\{\begin{array}{l} \Delta^2\phi_j = \lambda_j\phi_j \text{ in } \Omega, \\ \phi_j = 0 \text{ on } \partial\Omega, \\ \Delta\phi_j = 0 \text{ on } \partial\Omega, \end{array}\right.$$

where $\lambda_j = \omega_j^2$.
The Jerison-Lebeau estimate can be written

$$\|u\|_{L^2(\Omega)} \leq Ce^{C\lambda^{1/4}}\|u\|_{L^2(\mathscr{O})}, \quad \lambda > 0, \quad u \in \text{Span}\{\phi_j;\ \lambda_j \leq \lambda\}.$$

A spectral inequality for bi-Laplace operator: clamped boundary conditions

There exist $(\varphi_j)_j$ an orthonormal basis of $L^2(\Omega)$ and $\mu_j > 0$, such that

$$\begin{cases} \Delta^2 \varphi_j = \mu_j \varphi_j \text{ in } \Omega, \\ \varphi_j = 0 \text{ on } \partial\Omega, \\ \partial_n \varphi_j = 0 \text{ on } \partial\Omega, \end{cases}$$

Theorem

Let $\mathscr{O}$ be an open subset of Ω. There exists $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C e^{C\mu^{1/4}} \|u\|_{L^2(\mathscr{O})}, \quad \mu > 0, \quad u \in \operatorname{Span}\{\varphi_j;\ \mu_j \leq \mu\}.$$

Step of proof for Laplace operator(1)

Three kind of Carleman estimates: there exist $C > 0$, $\tau_0 > 0$ such that for all $\tau \geq \tau_0$ and $w \in \mathscr{C}_0^\infty(V_j)$, (Here $\Delta = \partial_s^2 + \Delta_x$).

$$\text{In } V_1,\ \tau^{3/2} \| e^{\tau\varphi} w \|_{L^2(Z)} + \tau^{1/2} \| e^{\tau\varphi} w \|_{H^1(Z)} \lesssim \| e^{\tau\varphi} \Delta w \|_{L^2(Z)}.$$

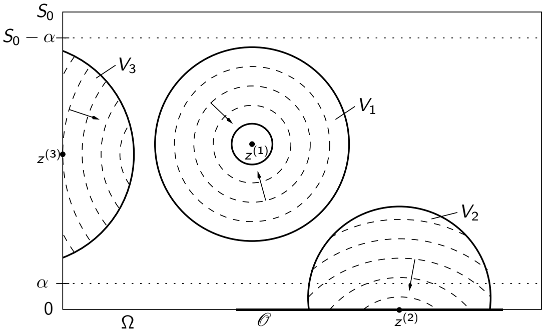


Figure: $Z = (0, S_0) \times \Omega$

Step of proof for Laplace operator(2)

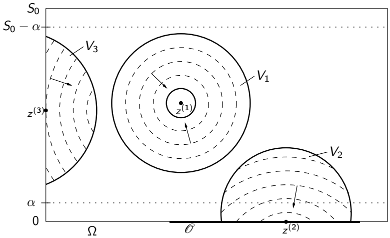
Three kind of Carleman estimates: there exist $C > 0$, $\tau_0 > 0$ such that for all $\tau \geq \tau_0$ and $w \in \mathscr{C}_0^\infty(V_j)$,

$$\text{In } V_2, \sum_{j=0,1} \tau^{3/2-j} \| e^{\tau\varphi} w \|_{H^j(Z)}$$

$$\lesssim \| e^{\tau\varphi} \Delta w \|_{L^2(Z)} + \tau^{1/2} \Big(| e^{\tau\varphi} w_{|s=0^+} |_{H^1_\tau(\mathscr{O})} | e^{\tau\varphi} \partial_s w_{|s=0^+} |_{L^2(\mathscr{O})} \Big).$$

$$\text{In } V_3, \sum_{j=0,1} \tau^{3/2-j} \| e^{\tau\varphi} w \|_{H^j(Z)}$$

$$\lesssim \| e^{\tau\varphi} \Delta w \|_{L^2(Z)} + \tau^{1/2} | e^{\tau\varphi} w_{|\partial\Omega} |_{H^1_\tau((\alpha, S_0-\alpha)\times\partial\Omega)},$$

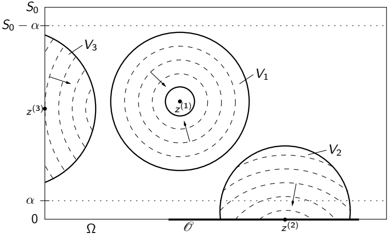


Interpolation estimates

$$\|v\|_{H^1(B(z^{(1)},3r))} \lesssim \|v\|^{1-\delta}_{H^1(Z)} \Big(\|\Delta v\|_{L^2(Z)} + \|v\|_{H^1(B(z^{(1)},r))} \Big)^{\delta},$$

$$\|v\|_{H^1(V\cap Z)} \lesssim \|v\|^{1-\delta}_{H^1(Z)} \Big(\|\Delta v\|_{L^2(Z)} + \big|v_{|s=0^+}\big|_{H^1(\mathscr{O})} + \big|\partial_s v_{|s=0^+}\big|_{L^2(\mathscr{O})} \Big)^{\delta},$$

$$\|v\|_{H^1(V\cap Z)} \lesssim \|v\|^{1-\delta}_{H^1(Z)} \big(\|\Delta v\|_{L^2(Z)} + \|v\|_{H^1(\mathscr{Q})} \big)^{\delta}, \quad v_{|(0,S_0)\times\partial\Omega} = 0.$$



$$\|v\|_{H^1(\alpha,S_0-\alpha)} \lesssim \|v\|^{1-\delta}_{H^1(Z)} \Big(\|\Delta v\|_{L^2(Z)} + \big|v_{|s=0^+}\big|_{H^1(\mathscr{O})} + \big|\partial_s v_{|s=0^+}\big|_{L^2(\mathscr{O})} \Big)^{\delta}$$

Spectral estimate

Recall interpolation estimate.

$$\|v\|_{H^1(\alpha,S_0-\alpha)} \lesssim \|v\|_{H^1(Z)}^{1-\delta}\Big(\|\Delta v\|_{L^2(Z)} + |v_{|s=0^+}|_{H^1(\mathscr{O})} + |\partial_s v_{|s=0^+}|_{L^2(\mathscr{O})}\Big)^{\delta}$$

Let $u = \sum_{\omega_j \leq \omega} u_j \phi_j \in \mathrm{Span}\{\phi_j;\ \omega_j \leq \omega\}$, we apply interpolation estimate to $v(s,x)$, defined by

$$v(s,x) = \sum_{\omega_j \leq \omega} u_j \omega_j^{-1/2} \sinh(\omega_j^{1/2} s)\phi_j(x).$$

We have $v_{|s=0^+} = 0$, $\partial_s v_{|s=0^+} = \sum_{\omega_j \leq \omega} u_j \phi_j(x)$ and we can estimate $\|v\|_{H^1(\alpha,S_0-\alpha)}$ and $\|v\|_{H^1(Z)}^{1-\delta}$ resp. by below and by above to have

$$\|u\|_{L^2(\Omega)} \leq Ce^{C\omega^{1/2}}\|u\|_{L^2(\mathscr{O})},$$

Step for bi-Laplace operator

We introduce $P = \partial_s^4 + \Delta^2$.

- Three Carleman estimates in regions 1, 2 and 3.
- Interpolation estimates.

$$\|v\|_{H^3(\alpha, S_0-\alpha)} \leq C\|v\|_{H^3(Z)}^{1-\delta}\|\partial_s^3 v_{|s=0}\|_{L^2(\mathscr{O})}^{\delta}.$$

- For $u = \sum_{\mu_j \leq \mu} u_j \varphi_j$, let

$$v(s,.) = \sum_{\mu_j \leq \mu} u_j \mu_j^{-3/4} f(\mu_j^{1/4} s)\varphi_j,$$

where $f(0) = f'(0) = f''(0) = 0$, $f^{(3)}(0) = 1$ and $\partial_s^4 f = -f$.

The main new result is the first step.

Known result for bi-Laplace operator

$$\tau^{3/2}\|e^{\tau\varphi}w\|_{L^2(Z)} + \tau^{1/2}\|e^{\tau\varphi}w\|_{H^1(Z)} + \tau^{-1/2}\|e^{\tau\varphi}w\|_{H^2(Z)} \lesssim \|e^{\tau\varphi}\Delta w\|_{L^2(Z)}.$$

Applying this to $w = \Delta v$ we obtain

$$\tau^{3/2}\|e^{\tau\varphi}\Delta v\|_{L^2(Z)} + \tau^{1/2}\|e^{\tau\varphi}\Delta v\|_{H^1(Z)} + \tau^{-1/2}\|e^{\tau\varphi}\Delta v\|_{H^2(Z)} \lesssim \|e^{\tau\varphi}\Delta^2 v\|_{L^2(Z)}.$$

Using first Carleman estimate (with a shift in Sobolev exponent), we obtain

$$\tau^{3}\|e^{\tau\varphi}v\|_{L^2(Z)} + \tau^{2}\|e^{\tau\varphi}v\|_{H^1(Z)} + \tau\|e^{\tau\varphi}v\|_{H^2(Z)} + \|e^{\tau\varphi}v\|_{H^3(Z)} \lesssim \|e^{\tau\varphi}\Delta^2 v\|_{L^2(Z)}.$$

Problem with Carleman estimate in interior

$$\tau^3\|e^{\tau\varphi}v\|_{L^2(Z)} + \tau^2\|e^{\tau\varphi}v\|_{H^1(Z)} + \tau\|e^{\tau\varphi}v\|_{H^2(Z)} + \|e^{\tau\varphi}v\|_{H^3(Z)} \lesssim \|e^{\tau\varphi}\Delta^2 v\|_{L^2(Z)}.$$

- No large parameter in front of $\|e^{\tau\varphi}v\|_{H^3(Z)}$. Problems with perturbation by operator of order 3. ($\varphi = e^{\gamma\psi}$)
- Naive method but this estimate cannot be improved.
- This method cannot used to prove estimations at boundary, i.e. regions 2 and 3.
- We have to treat $\partial_s^4 + \Delta^2$ but same problem if $\varphi'_s = 0$.

Factorization and notations

We write $P = \partial_s^4 + \Delta^2 = P_1 P_2$ with $P_k = (-1)^k i D_s^2 + A$, $D_s = -i\partial_s$ and $A = -\Delta$.

We write $P_\varphi = e^{\tau\varphi} P e^{-\tau\varphi} = Q_1 Q_2$ with $Q_k = e^{\tau\varphi} P_k e^{-\tau\varphi}$.

The principal symbol of q_k, (in semi-classical sense) is given by

$$q_k(z, \zeta, \tau) = (-1)^k i(\sigma + i\hat{\tau}_\sigma)^2 + a(x, \xi + i\hat{\tau}_\xi),$$
$$\text{where } \hat{\tau}(z, \tau) = (\hat{\tau}_\xi, \hat{\tau}_\sigma) = \tau d\varphi \in \mathbb{R}^{d+1},$$

and $a(x, \xi)$ denotes the principal symbol of A.

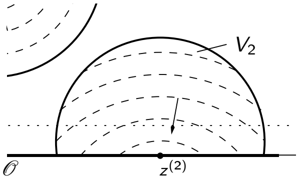
Carleman estimate in neighborhood $s = 0$

In this domain we have Carleman estimate without loss (i.e. $1/2$ derivative as usually).

$$\sum_{|\alpha|\leq 4} \tau^{7/2-|\alpha|} \|e^{\tau\varphi} D^{\alpha}_{s,x} u\|_{L^2(Z)}$$

$$\leq C\Big(\|e^{\tau\varphi}(\partial_s^4 + \Delta^2)u\|_{L^2(Z)} + \tau^{1/2} \sum_{j=0}^{3} |\operatorname{tr}(e^{\tau\varphi} D_s^j u)|_{0,3-j,\tau} \Big),$$

where $u \in \mathscr{C}_0^\infty(V_2)$.



Root properties: $s = 0$

The semi-classical characteristics set of a (pseudo-)differential operator A, with principal symbol $a(\varrho)$,

$\mathrm{char}(A) = \{\varrho = (z, \zeta, \tau) \in \overline{V} \times \mathbb{R}^N \times \mathbb{R}_+;\ (\zeta, \tau) \neq (0, 0), \text{ and } a(\varrho) = 0\}$,

Results for the characteristics sets of Q_k, $k = 1, 2$.

We have

$$\boxed{\mathrm{char}(Q_1) \cap \mathrm{char}(Q_2) = \emptyset}.$$

This means that the real roots cannot be double.

PROOF Let $\varrho = (z, \zeta, \tau)$ with $(\zeta, \tau) \neq (0, 0)$, be such that $q_1(\varrho) = q_2(\varrho) = 0$, that is
$(-1)^k i(\sigma + i\hat{\tau}_\sigma)^2 + a(x, \xi + i\hat{\tau}_\xi) = 0$, for both $k = 1$ and $k = 2$,

$$(\sigma + i\hat{\tau}_\sigma)^2 = 0, \quad a(x, \xi + i\hat{\tau}_\xi) = 0.$$

In particular this implies $\sigma = 0$ and $\hat{\tau}_\sigma = \tau \partial_s \varphi = 0$.

As here $\partial_s \varphi \neq 0$ we thus have $\sigma = \tau = 0$.
With $\tau = 0$, we have $\hat{\tau}_\xi = 0$, and thus $a(x, \xi) = 0$, implying $\xi = 0$.

Roots $s = 0$

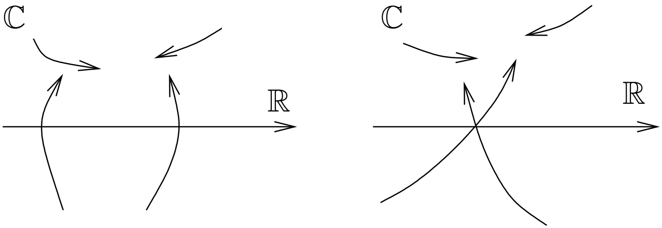


Figure: The right picture is forbidden

In this case we can apply the Bellassoued-Le Rousseau results.

Carleman estimate at $(0, S_0) \times \partial\Omega$

We assume $\partial\Omega = \{x_d = 0\}$.

Let $P = D_s^4 + A^2$. Let $z_0 = (s_0, x_0) \in (0, S_0) \times \partial\Omega$. Let $\varphi(z) = \varphi_{\gamma,\varepsilon}(z) = \exp(\gamma\psi(\varepsilon s, \varepsilon x', x_d))$. There exists an open neighborhood W of z_0 in $(0, S_0) \times \mathbb{R}^d$, $W \subset V$, and there exist $\tau_0 \geq \tau_*$, $\gamma_0 \geq 1$, $\varepsilon_0 \in (0, 1]$, and $C > 0$ such that

$$\gamma \sum_{|\alpha| \leq 4} \|\tilde{\tau}^{3-|\alpha|} e^{\tau\varphi} D_{s,x}^{\alpha} u\|_{L^2(Z)} + \sum_{0 \leq j \leq 3} |e^{\tau\varphi} D_{x_d}^{j} u_{|\partial Z}|_{7/2-j,\tilde{\tau}} \leq C\Big(\|e^{\tau\varphi} P u\|_{+} + \sum_{j=0,1} |e^{\tau\varphi} D_{x_d}^{j} u_{|\partial Z}|_{7/2-j,\tilde{\tau}}\Big),$$

for $\tau \geq \tau_0$, $\gamma \geq \gamma_0$, $\varepsilon \in [0, \varepsilon_0]$, and for $u = w_{|Z}$, with $w \in \mathscr{C}_0^{\infty}((0, S_0) \times \mathbb{R}^d)$ and $\operatorname{supp}(w) \subset W$.

Root properties at $(0, S_0) \times \partial\Omega$: notations

$$\hat{q}_k(z,\zeta,\hat{t}) = \left(\xi_d + i\hat{t}_{\xi_d}\right)^2 + (-1)^k i(\sigma + i\hat{t}_\sigma)^2 + r(x,\xi' + i\hat{t}_{\xi'}),$$

where $\hat{t} = \tau d\varphi_{\gamma,\varepsilon}$. But we consider $\hat{t}$ as an independent variable.

$$\hat{\mu}_k(z,\zeta',\hat{t}) := 4\hat{t}_{\xi_d}^2 \operatorname{Re}\hat{m}_k(z,\zeta',\hat{t}) - 4\hat{t}_{\xi_d}^4 + \left(\operatorname{Im}\hat{m}_k(z,\zeta',\hat{t})\right)^2,$$

where $\hat{m}_k(z,\zeta',\hat{t}) := (-1)^k i(\sigma + i\hat{t}_\sigma)^2 + r(x,\xi' + i\hat{t}_{\xi'})$.

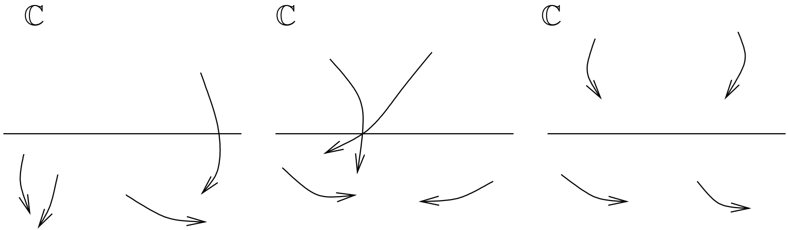
We denote by $\hat{\rho}_{k,+}(z,\zeta',\hat{t})$ and $\hat{\rho}_{k,-}(z,\zeta',\hat{t})$ the roots of $\hat{q}_k(z,\zeta,\hat{t})$ as polynomial with respect to ξ_d.

Root properties at $(0, S_0) \times \partial\Omega$

We want smooth roots (symbols) or roots in lower half space in $\mathbb{C}$.
We assume $\hat{t}_{\xi_d} \geq 0$

- $\operatorname{Im} \hat{\rho}_{k,-} \leq -\hat{t}_{\xi_d} \leq \operatorname{Im} \hat{\rho}_{k,+}$.
- $\hat{\rho}_{k,-} = \hat{\rho}_{k,+} \quad \Leftrightarrow \quad \hat{\rho}_{k,-} = \hat{\rho}_{k,+} = -i\hat{t}_{\xi_d} \quad \Leftrightarrow \quad \hat{m}_k = 0$.
- $\operatorname{Im} \hat{\rho}_{k,+} \lesseqqgtr 0 \quad \Leftrightarrow \quad \hat{\mu}_k \lesseqqgtr 0$.
- $\hat{\mu}_k(z, \zeta', \hat{t}) \geq C(|\hat{t}|^2 + |\zeta'|^2)^2$
  $\Rightarrow \operatorname{Im} \hat{\rho}_{k,+}(z, \zeta', \hat{t}) \geq C'(|\hat{t}|^2 + |\zeta'|^2)^{1/2}$.
- $|\hat{t}| \leq \theta_0|\zeta'|$, ($\theta_0$ sufficiently small) then the roots $\hat{\rho}_{k,\pm}$ are simple and non real, and moreover
  $\operatorname{Im} \hat{\rho}_{k,+} \geq C(|\hat{t}|^2 + |\zeta'|^2)^{1/2}, \quad \operatorname{Im} \hat{\rho}_{k,-} \leq -C(|\hat{t}|^2 + |\zeta'|^2)^{1/2}$
- if $\hat{\rho}_{k,+} \in \mathbb{R}$, $0 \leq \hat{t}_{\xi_d} \leq C\big(|\hat{t}'| + |\zeta'|\big)$, $|\zeta'| \leq C|\hat{t}|$, and $\operatorname{Im} \hat{\rho}_{k,-} = -2\hat{t}_{\xi_d}$.
- If $\hat{t}_{\xi_d} > 0$ and if $|\hat{t}'|/\hat{t}_{\xi_d}$ is sufficiently small, i.e. there exists $C_0, C_1 > 0$ such that if $|\hat{t}'| \leq C_0\hat{t}_{\xi_d}$ then
  $\hat{\rho}_{k,+} = \hat{\rho}_{k,-} \quad \Rightarrow \quad \operatorname{Im} \hat{\rho}_{k',\pm} \leq -C_1\hat{t}_{\xi_d}$, where $k \neq k'$.
- If$|\hat{t}'| \leq C_0\hat{t}_{\xi_d}$, $C_0 > 0$. There exists $\delta_0 > 0$, such that if $\hat{\mu}_k(z, \zeta', \hat{t}) \geq -\delta_0(|\hat{t}|^2 + |\zeta'|^2)^2$, the roots of $\hat{q}_k$ are simple.

Three regions



$\hat{\mu}_k \leq -\delta'(|\hat{t}|^2 + |\zeta'|^2)^2$ $\quad$ $\hat{\mu}_k \geq -\delta(|\hat{t}|^2 + |\zeta'|^2)^2$ $\quad$ $|\hat{t}| \leq C|\zeta'|$

for $k = 1, 2$

Notations

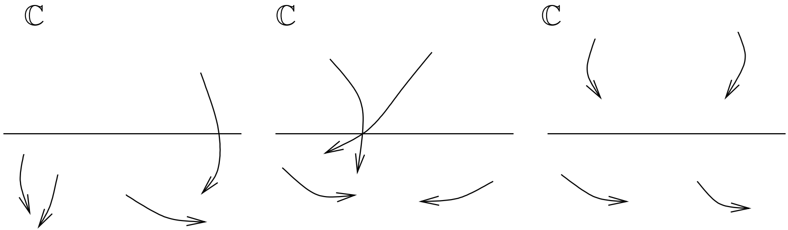
Norm notation $\|u\|_{m,\ell,\tilde{\tau}}$ means

- integration over $x_d > 0$.
- $\tilde{\tau} = \tau e^{\gamma\psi}$.
- m derivatives in all variables ($\tilde{\tau}^k D^\alpha$, $k + |\alpha| \leq m$ if $m \in \mathbb{N}$.)
- ℓ derivatives in tangential variables (s and x').

Norm notation $|u|_{m,\ell,\tilde{\tau}}$ means

- integration on boundary $x_d = 0$.
- estimation for $\tilde{\tau}^k D_{x'}^\alpha D_d^j u_{|x_d=0}$, $j \leq m$ and $|\alpha| + k + j \leq \ell$, if $\ell \in \mathbb{N}$. And $\langle D_{x'}, \tilde{\tau}\rangle^\ell D_d^j u_{|x_d=0}$ if ℓ is not an integer.

Three regions



$\hat{\mu}_k \leq -\delta'(|\hat{t}|^2 + |\zeta'|^2)^2$ $\qquad$ $\hat{\mu}_k \geq -\delta(|\hat{t}|^2 + |\zeta'|^2)^2$ $\qquad$ $|\hat{t}| \leq C|\zeta'|$

for $k = 1, 2$

Estimate in first region (1)

We have modulo error terms and cut-off in frequencies

$$\|w\|_{2,0,\tilde\tau} + |\operatorname{tr}(w)|_{1,1/2,\tilde\tau} \lesssim \|Q_1 w\|$$

$$\gamma^{1/2}\|\tilde\tau^{-1/2} v\|_{2,\ell,\tilde\tau} + |\operatorname{tr}(v)|_{1,\ell+1/2,\tilde\tau} \lesssim \|Q_2 v\|_{0,\ell,\tilde\tau} + |\operatorname{tr}(v)|_{0,\ell+3/2,\tilde\tau}.$$

We take $w = Q_2 u$ and we have

$$\|Q_2 u\|_{2,0,\tilde\tau} + |\operatorname{tr}(Q_2 u)|_{1,1/2,\tilde\tau} \lesssim \|Q_1 Q_2 u\|$$

$$\|Q_2 u\|_{2,0,\tilde\tau} \backsimeq \|D_{x_d}^2 Q_2 u\|_{0,0,\tilde\tau} + \|D_{x_d} Q_2 u\|_{0,1,\tilde\tau} + \|Q_2 u\|_{0,2,\tilde\tau}.$$

We take $v = D_{x_d}^k u$ and $\ell = 2-k$, $k = 0,1,2$

$$\gamma^{1/2}\|\tilde\tau^{-1/2} D_{x_d}^k u\|_{2,2-k,\tilde\tau} + |\operatorname{tr}(D_{x_d}^k u)|_{1,2-k+1/2,\tilde\tau} \lesssim \|Q_2 D_{x_d}^k u\|_{0,2-k,\tilde\tau} + |\operatorname{tr}(D_{x_d}^k u)|_{0,2-k+3/2,\tilde\tau}.$$

Summing on k,

$$\gamma^{1/2}\|\tilde\tau^{-1/2} u\|_{4,0,\tilde\tau} + |\operatorname{tr}(u)|_{3,1/2,\tilde\tau} \lesssim \|Q_2 u\|_{2,0,\tilde\tau} + |\operatorname{tr}(u)|_{2,3/2,\tilde\tau}$$

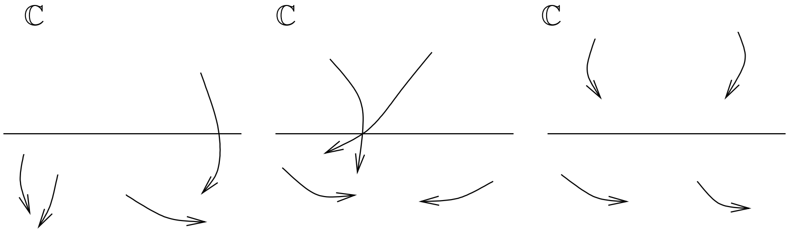
Estimate in first region (2)

Remind

$$\|Q_2 u\|_{2,0,\tilde{\tau}} + |\operatorname{tr}(Q_2 u)|_{1,1/2,\tilde{\tau}} \lesssim \|Q_1 Q_2 u\|.$$

$$\begin{aligned}
\gamma^{1/2}\|\tilde{\tau}^{-1/2} u\|_{4,0,\tilde{\tau}} + |\operatorname{tr}(u)|_{3,1/2,\tilde{\tau}} \\
&\lesssim \|Q_2 u\|_{2,0,\tilde{\tau}} + |\operatorname{tr}(u)|_{2,3/2,\tilde{\tau}} \\
&\lesssim \|Q_2 u\|_{2,0,\tilde{\tau}} + |\operatorname{tr}(u)|_{1,5/2,\tilde{\tau}} + |\operatorname{tr}(Q_2 u)|_{0,3/2,\tilde{\tau}} \\
&\lesssim \|Q_1 Q_2 u\| + |\operatorname{tr}(u)|_{1,5/2,\tilde{\tau}}.
\end{aligned}$$

Three regions



$\hat{\mu}_k \leq -\delta'(|\hat{t}|^2 + |\zeta'|^2)^2$ $\qquad$ $\hat{\mu}_k \geq -\delta(|\hat{t}|^2 + |\zeta'|^2)^2$ $\qquad$ $|\hat{t}| \leq C|\zeta'|$

for $k = 1, 2$

Estimate in second region

We have modulo error terms and cut-off in frequencies

$$\gamma \|\tilde{\tau}^{-1} v\|_{4,0,\tilde{\tau}} + |\operatorname{tr}(v)|_{3,1/2,\tilde{\tau}} \leq C\Big(\|Q_1 Q_2 v\|_{+} + |\operatorname{tr}(v)|_{1,5/2,\tilde{\tau}} \Big),$$

- We have a loss of one derivative ($\tilde{\tau}^{-1}$).
- This is compensate by a γ.
- We need the two first traces in the right hand side.

Some ideas of proof

In this region we have 4 smooth roots $\rho_{k,\pm}$ for $k = 1, 2$.
$\operatorname{Im} \rho_{k,-} < 0$ and we can have $\rho_{1,+} = \rho_{2,+} \in \mathbb{R}$.
We want follow the same way to prove estimate

$$\gamma^{1/2} \|\tilde{\tau}^{-1/2} v\|_{1,\ell,\tilde{\tau}} \lesssim \|(D_d - \mathrm{op}(\rho_{k,+}))v\|_{0,\ell,\tilde{\tau}} + |\operatorname{tr}(v)|_{0,\ell+1/2,\tilde{\tau}}$$

$$\gamma \|\tilde{\tau}^{-1} v\|_{2,\ell,\tilde{\tau}} \lesssim \|(D_d - \mathrm{op}(\rho_{1,+}))(D_d - \mathrm{op}(\rho_{2,+}))v\|_{0,\ell,\tilde{\tau}} + |\operatorname{tr}(v)|_{1,\ell+1/2,\tilde{\tau}}$$

$$\|w\|_{2,0,\tilde{\tau}} + |\operatorname{tr}(w)|_{1,1/2,\tilde{\tau}} \lesssim \|(D_d - \mathrm{op}(\rho_{1,-}))(D_d - \mathrm{op}(\rho_{2,-}))w\|_{0,0,\tilde{\tau}}$$

Taking $w = (D_d - \mathrm{op}(\rho_{1,+}))(D_d - \mathrm{op}(\rho_{2,+}))v$, we obtain

$$\gamma \|\tilde{\tau}^{-1} v\|_{4,0,\tilde{\tau}} + |\operatorname{tr}(v)|_{3,1/2,\tilde{\tau}} \lesssim \|Q_1 Q_2 v\|_+ + |\operatorname{tr}(v)|_{1,5/2,\tilde{\tau}},$$

We have to use symbolic calculus.
PROBLEM: the remainder terms have the same strength than the left hand side. To follow the order in $\tilde{\tau}$, γ and ε we introduce an adapted pseudo-differential calculus. ($\tilde{\tau} = \tau e^{\gamma\psi}$).