Photo-Acoustic imaging in layered media

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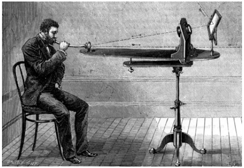
(joint works with **Kui Ren**, Texas at Austin, USA.)

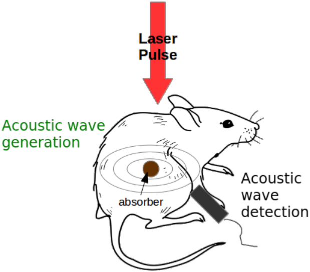
4 juillet 2016

Photo-Acoustic effect

Photo-Acoustic effect (Bell, 1880) : can we hear the light or can we see the sound ?



- Laser pulse



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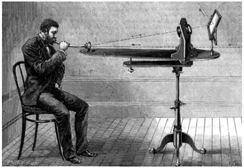

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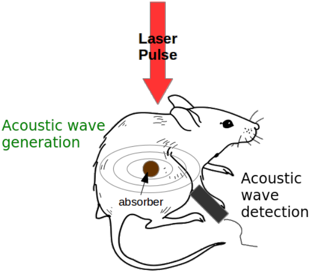

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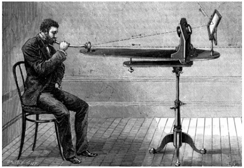

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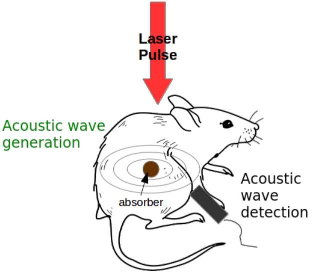

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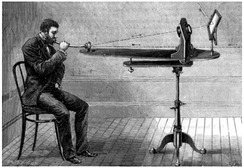

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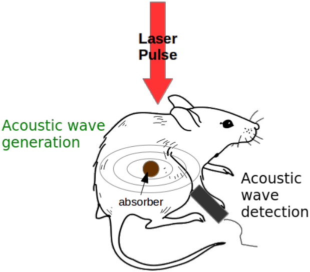

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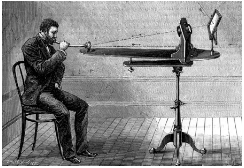

- Laser pulse
- Heat the absorbers
- Thermoelastic expansion of the absorbers
- Propagation of the acoustic wave in the tissue
- Detection of the acoustic waves at the boundary

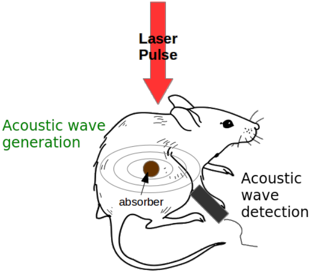

Photo-Acoustic effect

- In diffusive regime, optical radiation (laser) is modeled by :

$$\left\{ \begin{array}{lcll} -\nabla \cdot D_a(\mathbf{x})\nabla u + \mu_a(\mathbf{x}) u & = & 0 & \mathbf{x} \in \Omega \\ u & = & g & \mathbf{x} \in \partial\Omega \end{array} \right.$$

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$$\begin{cases} \frac{\partial^2 p(\mathbf{x},t)}{\partial t^2} - c^2(\mathbf{x})\Delta p(\mathbf{x},t) & = & 0 & \mathbf{x} \in \mathbb{R}^n, t \in \mathbb{R}_+ \\ p(\mathbf{x},0) & = & \color{red}{\mu_a(\mathbf{x})u(\mathbf{x})} & \mathbf{x} \in \mathbb{R}^n \\ \frac{\partial p}{\partial t}(\mathbf{x},0) & = & 0 & \mathbf{x} \in \mathbb{R}^n \end{cases}$$

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- Measurements : $p(x,t) \quad \mathbf{x} \in \partial\Omega, t \in (0,T]$.

Photo-Acoustic : inversion

Inversion :

$p(\mathbf{x}, t) \quad \mathbf{x} \in \partial\Omega, t \in (0, T] \longrightarrow (D_a(\mathbf{x}), \mu_a(\mathbf{x}), c(\mathbf{x}))\,, \ \mathbf{x} \in \Omega.$

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- Acoustic inversion (first step) :

$$p(\mathbf{x}, t), \quad \mathbf{x} \in \partial\Omega, t \in (0, T] \longrightarrow f(\mathbf{x}) = p(\mathbf{x}, 0), c(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

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- Optic inversion (second step) :

$$p(\mathbf{x},0) = \mu_a(\mathbf{x})u(\mathbf{x}), \quad \mathbf{x} \in \Omega \longrightarrow (D_a(\mathbf{x}), \mu_a(\mathbf{x}))\,, \quad \mathbf{x} \in \Omega,$$

where

$$\left\{ \begin{array}{lcll} -\nabla \cdot D_a(\mathbf{x})\nabla u + \mu_a(\mathbf{x})u & = & 0 & \mathbf{x} \in \Omega \\ u & = & g & \mathbf{x} \in \partial\Omega \end{array} \right.$$

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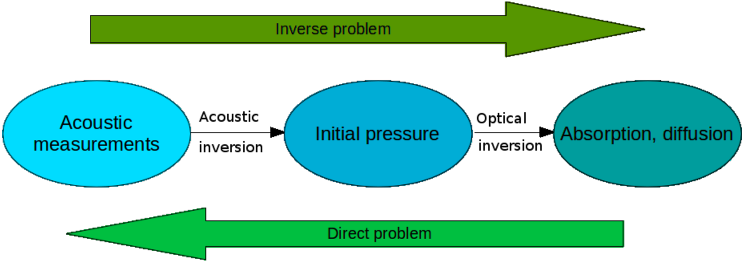

Two steps : acoustic inversion and optic inversion

Photo-Acoustic : the physicists point of view

Usually the physicists assume that $c(\mathbf{x})$ is constant and stop at the first step : why ?

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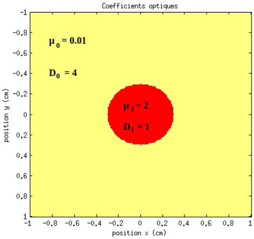
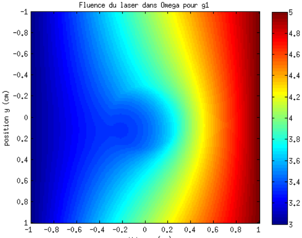
In fact the image $p(\mathbf{x}, 0) = \mu_a(\mathbf{x})u(\mathbf{x})$ has in general the same singularities as $\mu_a(\mathbf{x})$.

- μ_a, D_a et c are piecewise constants.
- $\Omega =]-1, 1[^2$ and measurements on $\partial\Omega \times (0, T)$.
- laser excitations : $g_1(x, y) = x + 4$.





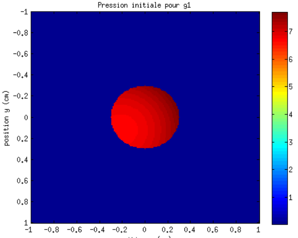
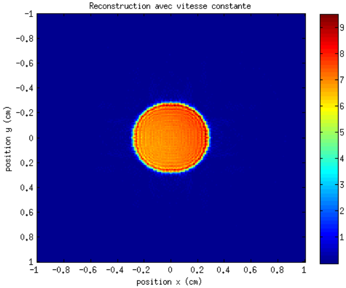


Photo-Acoustic : the physicists point of view

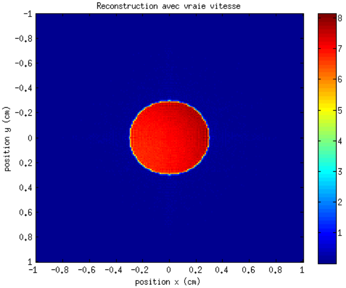
The drawbacks of the physicists approach :

- the approach also fails to provide the correct values of the absorption parameter.
- to distinguish between the singularities of the speed and the initial pressure

  Acoustic inversion with $g_1(x)$ and white noise of 5% :



L^2 error$= 0.20$



L^2 error $= 0.07$

Photo-Acoustic : the mathematicians point of view

Most of the mathematical works reconstruct the speed-initial pressure and the optical parameters separately.

- acoustic inversion : M. Bergounioux, P. Kuchment, L. Kunyansky, L. Qian, P. Stefanov, G. Uhlmann, ...
  - assuming the speed a constant, to determine the initial pressure
  - assuming the speed is known, to reconstruct the initial pressure
  - assuming the speed is close to a constant to recover the perturbation in the speed and the initial pressure
- optic inversion : G. Alessandrini, S. Arridge, H. Ammari, G. Bal, Y. Capdebosq, J. Schotland, O. Scherzer, Ber, K. Ren,...
  - assuming the internal data without critical points to recover the optical parameters
  - weighted stability estimates and general cases without assumption on the absence of critical points

Photo-Acoustic : the mathematicians point of view

acoustic inversion : Time Reversal -Neumann Series

Define Λ by $\Lambda p(\mathbf{x},0) = p(\mathbf{x},t)|_{\partial\Omega}$

$$\begin{cases} \frac{\partial^2 v(\mathbf{x},t)}{\partial t^2} - c^2(\mathbf{x})\Delta v(\mathbf{x},t) & = & 0 & \mathbf{x} \in \Omega, t \in (0,T) \\ v(\mathbf{x},T) & = & \varphi_T & x \in \Omega \\ \frac{\partial v}{\partial t}(\mathbf{x},T) & = & 0 & \mathbf{x} \in \Omega \\ v(\mathbf{x},t) & = & p(\mathbf{x},t) & x \in \partial\Omega, t \in (0,T), \end{cases}$$

where

$$\begin{cases} -\Delta\varphi_T(\mathbf{x}) & = & 0 & \mathbf{x} \in \Omega, \\ \varphi_T(\mathbf{x}) & = & p(\mathbf{x},T) & x \in \partial\Omega, \end{cases}$$

Define $Ap(\mathbf{x},t)|_{\partial\Omega} = v(\mathbf{x},0),\ \mathbf{x} \in \Omega$.

Theorem (Uhlmann-Stefanov)

Let Ω be non-trapping, and assume $T > T_0$. Then $A\Lambda = Id - K$, where K is a compact operator satisfying $\|K\| < 1$.

Photo-Acoustic : the mathematicians point of view

optic inversion : Unique Continuation

Assume that

$$H_j(\mathbf{x}) \quad = \quad \mu_a(\mathbf{x}) u_j(\mathbf{x}), \; \mathbf{x} \in \Omega \;\; j = 1, 2,$$

are given, where $u_j, j = 1, 2,$ are the laser intensities generated by two different illuminations $g_j, j = 1, 2$.

Theorem (Alessandrini 15)

If $\|H_j - \widetilde{H}_j\|_{L^2(\Omega)} \leq \varepsilon$ *and* $\|D_a - \widetilde{D}_a\|_{L^\infty(\partial\Omega)} \leq \varepsilon'$, *then*

$$\|D_a - \widetilde{D}_a\|_{L^2(\Omega)} + \|\mu_a - \widetilde{\mu}_a\|_{L^2(\Omega)} \leq C(\varepsilon + \varepsilon')^\delta,$$

where C *and* $\delta \in (0, 1)$ *do not depend on* ε *and* ε'.

Photo-Acoustic : the mathematicians point of view

The guidelines of the proof [G. Alessandrini and al. (2015)] :

- Let $V = \frac{u_2}{u_1} = \frac{H_2}{H_1}$ is a solution to [Bal-Uhlmann (2010)]

$$\left\{ \begin{array}{lcll} -\nabla \cdot \left(\overbrace{D_a(\mathbf{x}) u_1^2}^{\sigma(\mathbf{x})} \nabla V \right) & = & 0 & \mathbf{x} \in \Omega \\ V & = & \eta & \mathbf{x} \in \partial\Omega. \end{array} \right.$$

There exists $C > 0$ so that, for any $\sigma, \tilde{\sigma} \in \mathcal{E}_2$[G. Alessandrini (1986)],

$$\|(\sigma - \tilde{\sigma})|\nabla V|^2\|_{L^1(\Omega)} \leq C \left(\|V - \widetilde{V}\|_{L^2(\Omega)}^{1/3} + \|\sigma - \tilde{\sigma}\|_{L^\infty(\partial\Omega)} \right).$$

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- There exist $C > 0$ and $c > 0$, so that, for any $V \in \mathcal{S}$,

$$Cr^c \leq \|\nabla V\|_{L^2(B(\mathbf{x}_0, r))}, \;\; \mathbf{x}_0 \in \Omega^{r_0}, \; 0 < r < r_0.$$

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- $\frac{1}{u_1}$ is the solution to

$$\begin{cases} -\nabla \cdot (\sigma \nabla W) & = & -H_1 & x \in \Omega \\ W & = & \frac{1}{g_1} & \mathbf{x} \in \partial\Omega. \end{cases}$$

Photo-Acoustic penetration depth

The main limitation of PA imaging is the limited penetration depth when high resolution is desired.

For a resolution of $< 1mm$, only $5 \sim 6$ cm-penetration can be achieved.

FUJIFILM VisualSonics commercial products : photo-acoustic penetration depth is 1cm.

Nevertheless, PA imaging still provides excellent penetration-to-resolution ratio.

Recall that the optical resolution can be achieved with a penetration depth of only $1mm$!

A photo-acoustic imaging device

Photo-Acoustic : a realistic model

-Let $\Omega = (0, L) \times (0, H)$ be the domain occupied by the sample.
- Let $\Gamma_m = (0, L) \times \{y = H\}$ be the accessible part of the sample and $\Gamma_0 = (0, L) \times \{y = 0\}$.
- Assume $c(y) \geq c_m > 0$ to be a known smooth function that only depends on the vertical variable y.
-Assume $D_a(y) \geq D_0 > 0$ and $\mu_a(y) \geq \mu_0 > 0$ are smooth functions that depend only on the vertical variable y.

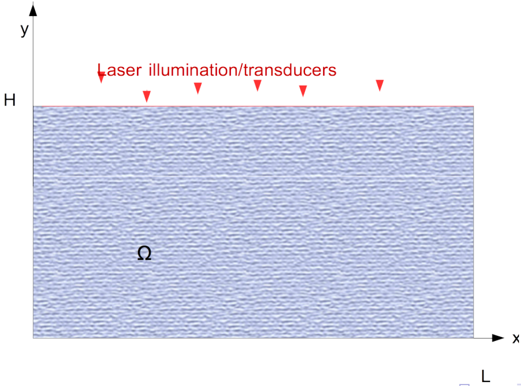

Photo-Acoustic : a realistic model

- The propagation of the optical wave in the sample is modeled by the following diffusion equation

$$\begin{cases} -\nabla \cdot D_a(y)\nabla u(\mathbf{x}) + \mu_a(y)u(\mathbf{x}) = 0 & \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \Gamma_m, \\ u(\mathbf{x}) = 0 & \mathbf{x} \in \Gamma_0, \\ u(0,y) = u(L,y) & y \in (0,H), \end{cases}$$

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- The pressure wave $p(\mathbf{x}, t)$ generated by the photoacoustic effect satisfies

$$\begin{cases} \partial_{tt} p(\mathbf{x}, t) = c^2(y)\Delta p(\mathbf{x}, t) & \mathbf{x} \in \Omega, t \geq 0, \\ \partial_\nu p(\mathbf{x}, t) + \beta \partial_t p(\mathbf{x}, t) = 0 & \mathbf{x} \in \Gamma_m, t \geq 0, \\ p(\mathbf{x}, t) = 0 & \mathbf{x} \in \Gamma_0, t \geq 0, \\ p((0, y), t) = p((L, y), t) & y \in (0, H), t \geq 0, \\ p(\mathbf{x}, 0) = f_0(\mathbf{x}) = \mu(y)u(\mathbf{x}),\ \partial_t p(\mathbf{x}, 0) = f_1(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases}$$

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- Measurements : $p(\mathbf{x}, t) \quad \mathbf{x} \in \Gamma_m, t \in (0, T]$.

Photo-Acoustic : a realistic model

Notice that in the first model of photo-acoustic imaging $f_0(\mathbf{x})$ is given by $\mu_a(y)u(\mathbf{x})$ and $f_1 = 0$ inside Ω.

Here $f_1(\mathbf{x})$ is considered as the correction of the photo-acoustic effect at the interface Γ_m, and is assumed to satisfy

$$\begin{cases} \Delta f_1(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ f_1(\mathbf{x}) = -\frac{1}{\beta}\partial_\nu f_0(\mathbf{x}) & \mathbf{x} \in \Gamma_m, \\ f_1(\mathbf{x}) = 0 & \mathbf{x} \in \Gamma_0, \\ f_1(0, y) = f_1(L, y) & y \in (0, H). \end{cases} \tag{1}$$

Assume that $c(y)$ **is given**, our objective is to reconstruct $\mu_a(y)$ and $D(y)$ for $y \in (0, H)$ from the data $p(\mathbf{x}, t)$, $\mathbf{x} \in \Gamma_m$, $t \in (0, T]$.

$$p(\mathbf{x}, t), \ \ \mathbf{x} \in \Gamma_m, \ t \in (0, T] \longrightarrow (\mu_a(y), D(y)), \ y \in (0, H).$$

Photo-Acoustic : a realistic model

Let

$$\mathcal{O}_M = \{(D,\mu) \in C^2([0,H])^2;\ \ D > D_0,\ \mu > \mu_0;\ \ \|\mu\|_{C^2}, \|D\|_{C^2} \le M\},$$

where $D_0 > 0, \mu_0 > 0$ and $M > \max(D_0, \mu_0)$ are fixed real constants.

Let $\varphi_k(x), k \in \mathbb{N}$, be the Fourier orthonormal real basis of $L^2(0,L)$ satisfying $-\varphi_k''(y) = \lambda_k^2 \varphi_k(y)$, with $\lambda_k = \frac{2k\pi}{L}$.

Let (D,μ_a), $(\widetilde{D}, \tilde{\mu}_a)$ in $\mathcal{O}_M$ and $c(y) \in W^{1,\infty}(0,H)$ with $0 < c_m \le c^{-2}(y)$ and set $\theta = \sqrt{\|c^{-2}\|_{L^\infty}}$.

Let k_i, $i = 1,2$ be two distinct integers, and denote u_{k_i}, $i = 1,2$ and $\tilde{u}_{k_i}$, $i = 1,2$ the solutions to the system for $g_i = \varphi_{k_i}, i = 1,2$, with coefficients (D,μ_a) and $(\widetilde{D}, \tilde{\mu}_a)$ respectively.

Assume that $D(H) = \widetilde{D}(H)$, $D'(H) = \widetilde{D}'(H)$, and $k_1 < k_2$, and k_1 is large enough.

Theorem (Ren-T)

Then for $T > 2\theta H$, $\delta_a \in (0, \frac{1}{8})$ and $\delta_d \in (0, \frac{1}{12})$, there exist two constants $C_a, C_d > 0$, such that the following stability estimates hold.

$$\int_0^H |\mu_a - \tilde{\mu}_a|(y)dy \leq C_a \left(\sum_{i=1}^{2} \int_0^T \left(\frac{C_M}{T - 2\theta H} + \beta \right) \|\partial_t p_i - \partial_t \tilde{p}_i\|^2_{L^2(\Gamma_m)} + \|\partial_x p_i - \partial_x \tilde{p}_i\|^2_{L^2(\Gamma_m)} dt \right)^{\delta_a},$$

and

$$\int_0^H |D - \widetilde{D}|(y)dy \leq C_d \left(\sum_{i=1}^{2} \int_0^T \left(\frac{C_M}{T - 2\theta H} + \beta \right) \|\partial_t p_i - \partial_t \tilde{p}_i\|^2_{L^2(\Gamma_m)} + \|\partial_x p_i - \partial_x \tilde{p}_i\|^2_{L^2(\Gamma_m)} dt \right)^{\delta_d},$$

where

$$C_M = He^{\int_0^H c^2(s)|\partial_y(c^{-2}(s))|ds}(c^{-2}(H) + \beta^2).$$

Photo-Acoustic : the first step

Theorem (Ren-T)

Assume that $c(y) \in W^{1,\infty}(0,1)$ *with* $0 < c_m \le c^{-2}(y)$*. Let* $\theta = \sqrt{\|c^{-2}\|_{L^\infty}}$ *and* $T > 2\theta H$*. Then*

$$\int_\Omega |\nabla f_0(\mathbf{x})|^2 d\mathbf{x} \le \left(\frac{C_M}{T-2\theta H} + \beta\right) \int_0^T \|\partial_t p(\mathbf{x},t)\|^2_{L^2(\Gamma_m)} dt + \int_0^T \|\partial_x p(\mathbf{x},t)\|^2_{L^2(\Gamma_m)} dt,$$

$$\int_\Omega c^{-2}(y)|f_1(\mathbf{x})|^2 d\mathbf{x} \le \left(\frac{C_M}{T-2\theta H} + \beta\right) \int_0^T \|\partial_t p(\mathbf{x},t)\|^2_{L^2(\Gamma_m)} dt + \int_0^T \|\partial_x p(\mathbf{x},t)\|^2_{L^2(\Gamma_m)} dt,$$

with

$$C_M = H e^{\int_0^H c^2(s)|\partial_y(c^{-2}(s))|ds}(c^{-2}(H) + \beta^2).$$

Photo-Acoustic : the first step

Since $f_j(\mathbf{x})$ is L-periodic in the y variable, it has the following discrete Fourier decomposition

$$f_j(x,y) \;=\; \sum_{k\in\mathbb{N}} f_{jk}(y)\varphi_k(x) \qquad (x,y)\in\Omega,$$

where $\varphi_k(x), k\in\mathbb{N}$, is the orthonormal real basis of $L^2(0,L)$.

The pressure is L-periodic in the y variable, it also has the following discrete Fourier decomposition

$$p(x,y,t) \;=\; \sum_{k\in\mathbb{N}} p_k(y,t)\varphi_k(x) \qquad (x,y)\in\Omega,\; t>0,$$

where $p_k(y,t)$ satisfy

$$\begin{cases} \frac{1}{c^2(y)}\partial_{tt}p(y,t) = \partial_{yy}p(y,t) - \lambda_k^2 p(y,t), & y\in(0,H), t\geq 0,\\ \partial_y p(H,t) + \beta\partial_t p(H,t) = 0 & t\geq 0,\\ p(0,t) = 0 & t\geq 0,\\ p(y,0) = f_{0k}(y),\; \partial_t p(y,0) = f_{1k}(y), & y\in(0,H), \end{cases}$$

Lemma

Let $\theta = \sqrt{\|c^{-2}\|_{L^\infty}}$ *and assume* $T > 2\theta H$*. Then*

$$\lambda_k^2 \int_0^H |f_{0k}(y)|^2 dy \le \left(\frac{C_M}{T - 2\theta H} + \beta \right) \int_0^T |\partial_t p_k(H,t)|^2 dt + \lambda_k^2 \int_0^T |p_k(H,t)|^2 dt,$$

$$\int_0^H c^{-2}(y)|f_{1k}(y)|^2 + |f'_{0k}(y)|^2 dy \le \left(\frac{C_M}{T - 2\theta H} + \beta \right) \int_0^T |\partial_t p_k(H,t)|^2 dt + \lambda_k^2 \int_0^T |p_k(H,t)|^2 dt,$$

for $k \in \mathbb{N}^*$*.*

The proof is based on techniques developed in R. Dager and E. Zuazua. Wave propagation, observation and control in 1d flexible multi-structures. Springer Science & Business Media, 2006.

Photo-Acoustic : the second step

Since $u(\mathbf{x})$ is L-periodic in the y variable, it has the following discrete Fourier decomposition

$$u(x, y) = \sum_{k \in \mathbb{N}} u_k(y)\varphi_k(x) \qquad (x, y) \in \Omega,$$

where $\varphi_k(x), k \in \mathbb{N}$, is the orthonormal real basis of $L^2(0, L)$. The real function $u_k(y)$ satisfies the following one dimensional elliptic equation

$$\begin{cases} -(D(y)u'(y))' + (\mu_a(y) + \lambda_k^2 D(y))u(y) = 0 \quad y \in (0, H), \\ u(H) = g_k, \quad u(0) = 0, \end{cases}$$

where g_k is the Fourier coefficient of g in the same basis, that is

$$g(x) = \sum_{k \in \mathbb{N}} g_k\varphi_k(x) \qquad x \in (0, L).$$

Photo-Acoustic : the second step

Let $h_i(y) = \mu_a(y) u_{k_i}(y),$, and $h_i(y) = \tilde{\mu}_a(y)\tilde{u}_{k_i}(y),\ i = 1,2.$

Theorem (Ren-T)

For $\delta_a \in (0, \frac{1}{4})$ and $\delta_d \in (0, \frac{1}{6})$, there exist two constants $C_a > 0$ and and $C_d > 0$ that only depends on $(\mu_0, D_0, k_1, k_2, M, L, H)$ respectively such that the following stability estimates hold.

$$\int_0^H |\mu_a - \tilde{\mu}_a|(y)dy \leq C_a \left(\|h_1 - \tilde{h}_1\|_{C^1} + \|h_2 - \tilde{h}_2\|_{C^1} \right)^{\delta_a},$$

and

$$\int_0^H |D - \widetilde{D}|(y)dy \leq C_d \left(\|h_1 - \tilde{h}_1\|_{C^1} + \|h_2 - \tilde{h}_2\|_{C^1} \right)^{\delta_d}.$$

Conclusion and remarks

We derived global stability estimates for the reconstruction of the optical parameters from measurement of the acoustic waves on the boundary.
The results show that the two steps of the inversion are stable if we consider parameters that only depend on the vertical variable to the boundary where the measurements are taken.

- to reconstruct also the speed $c(y)$ (we need further measurements).
- to generalize the results to dimension three (it can be reduced to a 2D problem).
- to consider the case where β is a function of x (supported at the position of the transducers).
- to propose an algorithm that solve both inversions.

Thanks !