State estimation for linear age-structured population diffusion models

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The problem in a glance

A classical model for age-space structured populations is given by

$$\begin{cases} \dfrac{\partial p}{\partial t}(a,x,t) + \dfrac{\partial p}{\partial a}(a,x,t) & \\ \quad = -\mu(a)p(a,x,t) + k\Delta p(a,x,t), & a \in (0,a^*),\ x \in \Omega,\ t > 0, \\ p(a,x,t) = 0, & a \in (0,a^*),\ x \in \partial\Omega,\ t > 0, \\ p(a,x,0) = p_0(a,x), & a \in (0,a^*),\ x \in \Omega, \\ p(0,x,t) = \displaystyle\int_0^{a^*} \beta(a)p(a,t,x)\,\mathrm{d}a, & x \in \Omega,\ t > 0. \end{cases}$$

- $p(a,x,t)$: distribution density of the population of age a at spatial position x at time t;
- a^* : maximal life expectancy;
- k : diffusion coefficient;
- $\mu(a), \beta(a)$: death and birth rates (independent of x);

The problem in a glance

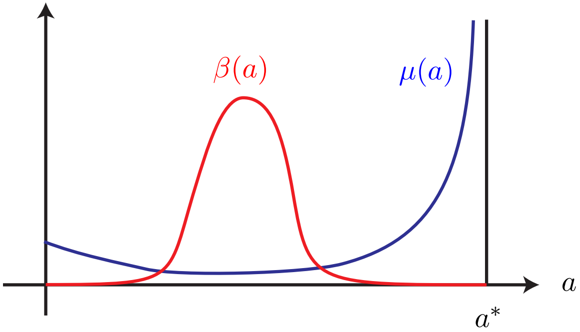
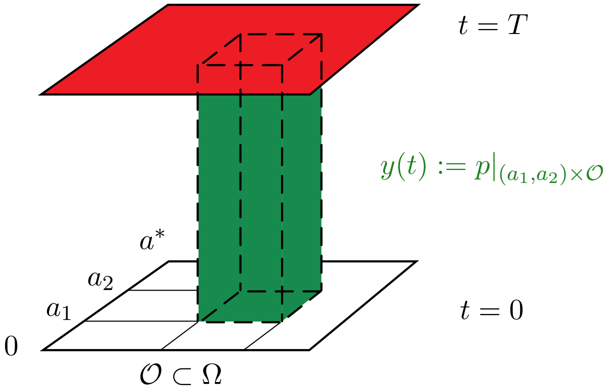


Figure: Typical birth and death rates.

The problem in a glance



Estimation problem

Knowing the output $y(t) := p|_{(a_1,a_2)\times\mathcal{O}}$ (but assuming that p_0 is unknown), estimate $p(a, x, T)$ for all $a \in (0, a^*)$ and $x \in \Omega$, as $T \to +\infty$.

The problem in a glance

$$\begin{cases} \dot{p}(t) = Ap(t), & t \in (0,T) \\ p(0) = p_0, \\ y(t) = Cp(t), & t \in (0,T), \end{cases}$$

where $C \in \mathcal{L}(X, Y)$, $Y := L^2((a_1, a_2) \times \mathcal{O})$ is defined by

$$C\varphi := \varphi|_{(a_1,a_2)\times\mathcal{O}} \text{ for all } \varphi \in X.$$

We introduce the Luenberger observer

$$\begin{cases} \dot{\hat{p}}(t) = A\hat{p}(t) + L\left(C\hat{p}(t) - y(t)\right), & t \in (0,T) \\ \hat{p}(0) = 0, \end{cases}$$

where $L \in \mathcal{L}(Y, X)$ is a linear operator to be defined.
Then the error $e := \hat{p} - p$ satisfies

$$\begin{cases} \dot{e}(t) = (A + LC)e(t), & t \in (0,T) \\ e(0) = -p_0. \end{cases}$$

The problem in a glance

Goal

Find L such that $e^{t(A+LC)}$ exponentially stable (detectability).

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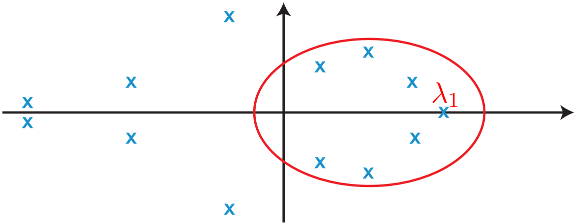
Find L such that $e^{t(A+LC)}$ exponentially stable (detectability).

How?



Spectrum of A : an infinite number of stable modes and a finite number of unstable modes.

Design an infinite dimensional Luenberger observer via a finite dimensional stabilizing operator.

Selective bibliography

- **Population Dynamics**
  - **Semigroup properties:** Song et al., Chan, Guo, Li et al., Langlais, Walker
  - **Controllability problems:** Ainseba, Anita, Iannelli, Langlais, Echarroudi, Maniar, Traoré, Kavian
  - **Inverse problems:** Traoré, Rundell, Di Blasio, Lorenzi, Perasso, Picart
  - **Numerical aspects:** Lopez, Trigiante, Milner, Kim, Huyer, Ayati, Dupont, Pelovska, Gerardo-Giorda
- **State Space Splitting**
  - **Abstract setting:** Russel, Triggiani, Jacobson & Nett, Jacob & Zwart
  - **Stabilization of PDE:** Barbu & Triggiani, Raymond et al., Badra & Takahashi

Outline

Outline

The model

$$\begin{cases} \dfrac{\partial p}{\partial t}(a,x,t) = -\dfrac{\partial p}{\partial a}(a,x,t) & \\ \quad -\mu(a)p(a,x,t) + k\Delta p(a,x,t), & a \in (0,a^*),\ x \in \Omega,\ t > 0, \\ p(a,x,t) = 0, & a \in (0,a^*),\ x \in \partial\Omega,\ t > 0, \\ p(a,x,0) = p_0(a,x), & a \in (0,a^*),\ x \in \Omega, \\ p(0,x,t) = \displaystyle\int_0^{a^*} \beta(a)p(a,t,x)\,\mathrm{d}a, & x \in \Omega,\ t > 0. \end{cases}$$

Assumptions

Typical assumptions on the birth and death rates β and μ:

- $\beta \in L^\infty(0, a^*)$, $\beta \geqslant 0$ a.e. in $(0, a^*)$;
- $\mu \in L^1_{\text{loc}}(0, a^*)$, $\mu \geqslant 0$ a.e. in $(0, a^*)$ and

$$\lim_{a \to a^*} \int_0^a \mu(s)\, \mathrm{d}s = +\infty.$$

We also introduce the function

$$\Pi(a) := \exp\left(- \int_0^a \mu(s)\, \mathrm{d}s \right)$$

which represents the probability to survive at age $a > 0$.
In particular

$$\lim_{a \to a^*} \Pi(a) = 0.$$

First order formulation

We introduce the Hilbert space $X := L^2\left((0,a^*)\times\Omega\right)$ and let A be defined by:

$$\mathcal{D}(A) = \Big\{ \varphi \in X \cap L^2\left((0,a^*), H_0^1(\Omega)\right) \Big| -\frac{\partial \varphi}{\partial a} - \mu\varphi + k\Delta\varphi \in X;$$

$$\varphi(a,\cdot)|_{\partial\Omega} = 0 \text{ for almost all } a \in (0,a^*);$$

$$\varphi(0,x) = \int_0^{a^*} \beta(a)\varphi(a,x)\,\mathrm{d}a \text{ for almost all } x \in \Omega \Big\}$$

$$A\varphi = -\frac{\partial \varphi}{\partial a} - \mu\varphi + k\Delta\varphi, \qquad \forall \varphi \in \mathcal{D}(A).$$

Well-posedness

The population dynamics problem reads then

$$\begin{cases} \dot{p}(t) = Ap(t), & t > 0 \\ p(0) = p_0. \end{cases}$$

Theorem (Chan and Guo, 1989)

- *A is the infinitesimal generator of a C_0-semigroup e^{tA} on X.*
- *If $p_0 \in X$, there exists a unique solution $p \in C([0,\infty), X)$.*
- *If $p_0 \in \mathcal{D}(A)$, there exists a unique solution $p \in C([0,\infty), \mathcal{D}(A)) \cap C^1([0,\infty), X)$.*

Diffusion free model

McKendrick–Von Foerster model (1959) describes the diffusion free case ($k = 0$) :

$$\begin{cases} \dfrac{\partial p}{\partial t}(a,t) = -\dfrac{\partial p}{\partial a}(a,t) - \mu(a)p(a,t), & a \in (0,a^*),\ t > 0, \\ p(a,0) = p_0(a), & a \in (0,a^*), \\ p(0,t) = \displaystyle\int_0^{a^*} \beta(a)p(a,t)\,\mathrm{d}a, & t > 0. \end{cases}$$

Diffusion free model

The population operator A_0 corresponding to the above system is defined as follows

$$\mathcal{D}(A_0) = \left\{ \varphi \in L^2(0,a^*) \,\middle|\, -\frac{\mathrm{d}\varphi}{\mathrm{d}a} - \mu\varphi \in L^2(0,a^*); \right.$$
$$\left. \varphi(0) = \int_0^{a^*} \beta(a)\varphi(a)\,\mathrm{d}a \right\}.$$

$$A_0\varphi = -\frac{\mathrm{d}\varphi}{\mathrm{d}a} - \mu\varphi, \qquad \forall\varphi \in \mathcal{D}(A_0).$$

Then the McKendrick–Von Foerster model reads then

$$\left\{ \begin{array}{ll} \dot{p}(t) = A_0 p(t), & t > 0 \\ p(0) = p_0. & \end{array} \right.$$

Diffusion free model

Theorem (Song et al., 1982)

1. A_0 *has compact resolvent and its spectrum is constituted of a countable (infinite) set of isolated eigenvalues with finite algebraic multiplicity.*
2. *The eigenvalues* $(\lambda_n^0)_{n\geqslant 1}$ *of* A_0 *(counted without multiplicity) the (complex) solutions of the characteristic equation*

$$F(\lambda) := \int_0^{a^*} \beta(a)\Pi(a)e^{-\lambda a}\,\mathrm{d}a = 1.$$

3. *The eigenvalues* $(\lambda_n^0)_{n\geqslant 1}$ *are of geometric multiplicity one:*

$$\varphi_n^0(a) = e^{-\lambda_n^0 a}\Pi(a) = e^{-\lambda_n^0 a - \int_0^a \mu(s)\,\mathrm{d}s}.$$

4. *Every vertical strip of the complex plane contains a finite number of eigenvalues of* A_0.

Diffusion free model

Theorem (Song et al., 1982)

The operator A_0 has a unique real eigenvalue λ_1^0. Moreover:

1. λ_1^0 *is of algebraic multiplicity one;*
2. $\lambda_1^0 > 0$ *(*< 0*)* $\Longleftrightarrow$ $F(0) = \int_0^{a^*} \beta(a)\Pi(a)\,\mathrm{d}a > 1$ *(*< 1*);*
3. λ_1^0 *is a real dominant eigenvalue:*

$$\lambda_1^0 > \mathrm{Re}\,(\lambda_n^0), \qquad \forall n \geqslant 2.$$

Back to the problem with diffusion

$$\mathcal{D}(A) = \left\{ \varphi \in X \cap L^2\left((0,a^*), H_0^1(\Omega)\right) \,\middle|\, -\frac{\partial \varphi}{\partial a} - \mu\varphi + k\Delta\varphi \in X; \right.$$

$$\varphi(a,\cdot)|_{\partial\Omega} = 0 \text{ for almost all } a \in (0,a^*);$$

$$\left. \varphi(0,x) = \int_0^{a^*} \beta(a)\varphi(a,x)\,\mathrm{d}a \text{ for almost all } x \in \Omega \right\}$$

$$A\varphi = -\partial_a\varphi - \mu\varphi + k\Delta\varphi, \qquad \forall \varphi \in \mathcal{D}(A).$$

Let $0 < \lambda_1^D < \lambda_2^D \leqslant \lambda_3^D \leqslant \cdots$ be the increasing sequence of eigenvalues of $-k\Delta$ with Dirichlet boundary conditions and let $(\varphi_n^D)_{n\geqslant 1}$ be a corresponding orthonormal basis of $L^2(\Omega)$.

Spectral properties

Theorem (Chan and Guo, 1989)

1. *A has compact resolvent and its (pure point) spectrum is*

$$\sigma(A) = \left\{ \lambda_i^0 - \lambda_j^D \, | i, j \in \mathbb{N}^* \right\}$$

2. *The eigenspace associated to an eigenvalue λ of A is given by*

$$\mathsf{Span} \left\{ \varphi_i^0(a)\varphi_j^D(x) = e^{-\lambda_i^0 a}\Pi(a)\varphi_j^D(x) \,\middle|\, \lambda_i^0 - \lambda_j^D = \lambda \right\}.$$

3. *The real eigenvalue λ_1 of A is dominant:*

$$\lambda_1 = \lambda_1^0 - \lambda_1^D > \mathrm{Re}\,(\lambda), \qquad \forall \lambda \in \sigma(A),\ \lambda \neq \lambda_1.$$

4. *λ_1 is a simple eigenvalue, the corresponding eigenspace being generated by*

$$\varphi_1(a, x) := \varphi_1^0(a)\varphi_1^D(x) = e^{-\lambda_1^0 a}\Pi(a)\varphi_1^D(x).$$

Spectral properties

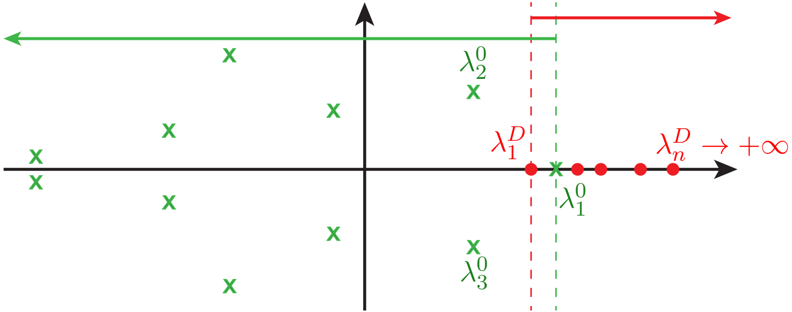


Figure: Spectra of A_0 and $-k\Delta$.

In this example, there is only 1 unstable eigenvalue λ_1:

$$\mathrm{Re}\,(\lambda_n^0) < \lambda_1^D < \lambda_1^0 < \lambda_2^D, \qquad \forall n \geqslant 2$$

$$\Longrightarrow \qquad \lambda_1 = \lambda_1^0 - \lambda_1^D > 0, \qquad \mathrm{Re}\,(\lambda_n) < 0, \qquad \forall n \geqslant 2$$

Compactness & Stability

Proposition (Chan and Guo, 1989)

The semigroup e^{tA} generated on X by A is compact for $t \geqslant a^*$.

This implies in particular that (see Zabczyk, 1975)

$$\omega_a(A) = \omega_0(A)$$

where $\omega_a(A) := \lim_{t \to +\infty} t^{-1} \ln \|e^{tA}\|$ denotes the **growth bound** of e^{tA} and $\omega_0(A) := \sup \{\operatorname{Re} \lambda \mid \lambda \in \sigma(A)\}$ the **spectral bound** of A.

Consequence

The above condition ensures that the exponential stability of e^{tA} is equivalent to the condition

$$\omega_0(A) = \sup \{\operatorname{Re} \lambda \mid \lambda \in \sigma(A)\} < 0.$$

Outline

Abstract framework

Consider

- $A : \mathcal{D}(A) \to X$ with compact resolvent on a Hilbert space X generating a C_0-semigroup in X,
- $C \in \mathcal{L}(X, Y)$, where Y is another Hilbert space.

We assume

(A1) A admits M eigenvalues (counted without multiplicities) with real part greater or equal than 0:

$$\cdots \leqslant \operatorname{Re}\lambda_{M+2} \leqslant \operatorname{Re}\lambda_{M+1} < 0 \leqslant \operatorname{Re}\lambda_M \leqslant \cdots \leqslant \operatorname{Re}\lambda_2 \leqslant \operatorname{Re}\lambda_1.$$

(A2) We have the equality

$$\omega_a(A) = \omega_0(A).$$

Detectability

Definition

The pair (A, C) is detectable if there exists $L \in \mathcal{L}(Y, X)$ such that $(A + LC)$ generates an exponentially stable semigroup.

We are going to show that:

Spectral observability of unstable eigenfunctions of A

$$\big(A\varphi = \lambda\varphi \text{ for } \lambda \in \Sigma_+ \text{ and } C\varphi = 0\big) \implies \varphi = 0$$

$$\Downarrow$$

Detectability of the finite dimensional system (A^+, C^+)

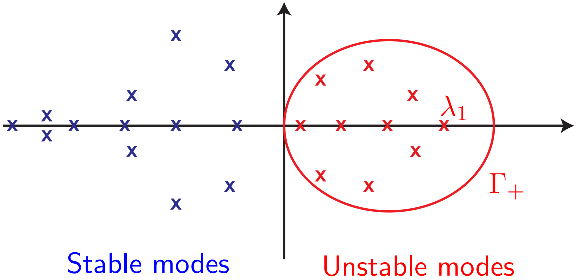
$$\Downarrow$$

Detectability of the infinite dimensional system (A, C)

Projection operator

We set $\Sigma_+ := \{\lambda_1, \ldots, \lambda_M\}$ and let Γ_+ be a positively oriented curve enclosing Σ_+ but no other point of the spectrum of A. Let $P_+ : X \to X$ be the projection operator defined by

$$P_+ := -\frac{1}{2\pi i} \int_{\Gamma_+} (\xi - A)^{-1} \, d\xi.$$



Splitting

We set $X_+ := P_+ X$ and $X_- := (I - P_+)X$, and then P_+ provides the following decomposition of X

$$X = X_+ \oplus X_- .$$

Following Russell and Triggiani, we can decompose our system into two subsystems :

- a finite dimensional system to be stabilized,
- a stable infinite dimensional system.

More precisely, X_+ and X_- are invariant subspaces under A (since A and P_+ commute) and the spectra of the restricted operators $A\big|_{X_+}$ and $A\big|_{X_-}$ are respectively Σ_+ and $\Sigma_- := \sigma(A) \setminus \Sigma_+$. We also define

$$A_+ := A|_{\mathcal{D}(A) \cap X_+} : \mathcal{D}(A) \cap X_+ \longrightarrow X_+,$$

$$A_- := A|_{\mathcal{D}(A) \cap X_-} : \mathcal{D}(A) \cap X_- \longrightarrow X_- .$$

Splitting

If A is **diagonalizable**, the space $X_+ = P_+ X$ is the finite dimensional space spanned by the eigenfunctions of A associated to the unstable eigenvalues:

$$X_+ = \bigoplus_{k=1}^{M} \mathrm{Ker}\,(A - \lambda_k).$$

and

$$\dim X_+ = \sum_{k=1}^{M} m_k^{\mathrm{G}}.$$

where $m_k^{\mathrm{G}} := \dim \mathrm{Ker}\,(A - \lambda_k)$ is the geometric multiplicity of λ_k.

Splitting

In the general case, the space X_+ is the finite dimensional space spanned by the generalized eigenfunctions of A associated to the unstable eigenvalues:

$$X_+ = \bigoplus_{k=1}^{M} \operatorname{Ker}\,(A - \lambda_k)^{m_k^{\mathrm{P}}}$$

where m_k^{P} is the multiplicity of the pole λ_k in the resolvent $(A - \lambda)^{-1}$.
The space $\operatorname{Ker}\,(A - \lambda_k)^{m_k^{\mathrm{P}}}$ is called the generalized eigenspace associated to λ_k. Its dimension m_k^{A} is the algebraic multiplicity of λ_k.

$$\dim X_+ = \sum_{k=1}^{M} m_k^{\mathrm{A}}.$$

Detectability result : from L_+ to L

Theorem

Let

- $Q_+ : Y \to Y_+ := CX_+$ *be the orthogonal projection operator from* Y *to* Y_+*,*
- $i_{X_+} : X_+ \to X$ *be the embedding operator from* X_+ *into* X*.*

Set

$$C_+ = Ci_{X_+} \in \mathcal{L}(X_+, Y_+)$$

and assume that the finite dimensional projected system (A_+, C_+) *is detectable through* $L_+ \in \mathcal{L}(Y_+, X_+)$*.*
Then, the infinite dimensional system (A, C) *is detectable through*

$$L = i_{X_+} L_+ Q_+ \in \mathcal{L}(Y, X).$$

Proof

For $L \in \mathcal{L}(Y, X)$, consider the system

$$\dot{z}(t) = (A + LC)z(t).$$

If we write $z = z_+ + z_-$ where $z_+ := P_+ z$ and $z_- := (I - P_+)z$, by applying P_+ and $(I - P_+)$ to the above equation, we obtain a corresponding splitting of the system into two subsystems:

$$\begin{cases} \dot{z}_+(t) &= A_+ z_+(t) + P_+ LCz(t), \\ \dot{z}_-(t) &= A_- z_-(t) + (I - P_+)LCz(t). \end{cases}$$

Taking $L = i_{X_+} L_+ Q_+$ and using the identities $P_+ i_{X_+} = \mathrm{Id}_{X_+}$ and $(I - P_+)i_{X_+} = 0$, we obtain

$$\begin{cases} \dot{z}_+(t) &= A_+ z_+(t) + L_+ Q_+ Cz(t), \\ \dot{z}_-(t) &= A_- z_-(t). \end{cases}$$

Proof

It follows from assumption **(A2)** that z_- is exponentially stable:

$$\|z_-(t)\| \leqslant K e^{-\omega_- t} \|z_-(0)\|$$

where $0 < \omega_- \leqslant -\mathrm{Re}\,\lambda_{M+1}$. On the other hand, by using $C_+ = Q_+ C i_{X_+}$ and since $i_{X_+} z_+ = z_+$, we have

$$\begin{aligned}
\dot{z}_+(t) &= A_+ z_+(t) + L_+ Q_+ C(z_+(t) + z_-(t)) \\
&= A_+ z_+(t) + L_+ Q_+ C i_{X_+} z_+(t) + L_+ Q_+ C z_-(t) \\
&= (A_+ + L_+ C_+) z_+(t) + L_+ Q_+ C z_-(t).
\end{aligned}$$

Proof

Using Duhamel's formula, we get

$$z_+(t) = \mathbb{T}_t^+ z_+(0) + \int_0^t \mathbb{T}_{t-s}^+ L_+ Q_+ C z_-(s) ds,$$

where $\mathbb{T}_t^+$ is the semigroup generated by $A_+ + L_+ C_+$, which is exponentially stable by the detectability assumption, i.e. there exists $\omega_+ > 0$ such that

$$\left\| \mathbb{T}_t^+ x \right\| \leqslant K e^{-\omega_+ t} \|x\| \qquad \forall x \in X_+, \forall t > 0.$$

Combined with exponential stability of z_-, this yields

$$\|z_+(t)\| \leqslant K \left\{ e^{-\omega_+ t} \|z_+(0)\| + \|L_+\| \|C\| \int_0^t e^{-\omega_+(t-s)} e^{-\omega_- s} \|z_-(0)\| ds \right\},$$

and consequently

$$\|z_+(t)\| \leqslant K \left(e^{-\omega_+ t} + \|L_+\| \|C\| \frac{e^{-\omega_+ t} - e^{-\omega_- t}}{\omega_- - \omega_+} \right) \|z_0\|.$$

Proof

It is then sufficient to choose ω_+ small enough such that $0 < \omega_+ < \omega_-$ to have the exponential decay of $t \mapsto z_+(t)$:

$$\|z_+(t)\| \leqslant K e^{-\omega_+ t} \|z_0\|, \qquad t > 0.$$

We have thus proved the exponential decay of $z = z_+ + z_-$. ■

Hautus test

The following result provide a sufficient condition of Hautus type for the detectability of the finite dimensional projected system (A_+, C_+).

Proposition

If the spectral observability condition (Hautus test)

$$\big(A\varphi = \lambda\varphi \text{ for } \lambda \in \Sigma_+ \text{ and } C\varphi = 0\big) \quad \Longrightarrow \quad \varphi = 0$$

is satisfied, then (A_+, C_+) is detectable.

Proof : Since $C_+ z_+ = C z_+$ for any $z_+ \in X_+$, if the Hautus test is satisfied, then it is clear that the following Hautus test is also satisfied:

$$\big(\ \varphi \in \mathcal{D}(A) \cap X_+ \mid A_+\varphi = \lambda\varphi \text{ and } C_+\varphi = 0\ \big) \quad \Longrightarrow \quad \varphi = 0.$$

As the above system is finite dimensional, (A_+, C_+) is detectable. ■

Remarks

Corollaire

If the Hautus test is satisfied, then (A, C) is detectable via the stabilizing output injection operator L defined previously.

- The matrices A_+ and C_+ are in practice of small size : their dimensions are respectively $\dim X_+ \times \dim X_+$ and $\dim Y_+ \times \dim X_+$.
- The stabilizing operator L_+ of the finite dimensional system (A_+, C_+) can be determined by solving a finite dimensional algebraic Riccati equation.

Outline

Observer

Assumptions **(A1)** ($M < \infty$ unstable eigenvalues) and **(A2)** ($\omega_a(A) = \omega_0(A)$) are satisfied for our population model and the problem of determining the stabilizing operator L for (A, C) fits into the framework described above.
It only remains to verify that the Hautus test is satisfied for our system (A, C):

Lemma

If $\varphi \in \mathcal{D}(A)$ satisfies $A\varphi = \lambda\varphi$ for $\lambda \in \Sigma_+$ and $C\varphi = 0$, then φ vanishes identically.

Proof of the spectral observability

Let λ be an unstable eigenvalue of A and let $\varphi \in \mathcal{D}(A)$ satisfying $A\varphi = \lambda\varphi$. Decomposing $\varphi(0,x)$ in the basis of $L^2(\Omega)$ constituted of the eigenfunctions of $-k\Delta$, the unique solution of the evolution system

$$\begin{cases} \dfrac{\partial \varphi}{\partial a}(a,x) = k\Delta\varphi(a,x) - (\lambda+\mu)\varphi(a,x), & a \in (0,a^*),\ x \in \Omega, \\ \varphi(a,x) = 0, & a \in (0,a^*),\ x \in \partial\Omega, \\ \varphi(0,x) = \displaystyle\sum_{j\in\mathbb{N}} \alpha_j \varphi_j^D(x), & x \in \Omega, \end{cases}$$

is given by

$$\varphi(a,x) = \sum_{j\in\mathbb{N}} \alpha_j e^{-(\lambda+\lambda_j^D)a}\Pi(a)\varphi_j^D(x).$$

Plugging the above expression in the renewal equation, we obtain

$$\sum_{j\in\mathbb{N}} \alpha_j \varphi_j^D(x) = \sum_{j\in\mathbb{N}} \alpha_j \left(\int_0^{a^*} \beta(a) e^{-(\lambda+\lambda_j^D)a}\Pi(a) \right) \varphi_j^D(x).$$

We see that is equivalent to, for any $j \in \mathbb{N}$, either $\alpha_j = 0$, either $\lambda + \lambda_j^D$ solves the characteristic equation of the diffusion free problem.

Proof of the spectral observability (end)

Consequently, we have

$$\varphi(a,x) = \sum_{j|\lambda+\lambda_j^D\in\sigma(A_0)} \alpha_j e^{-(\lambda+\lambda_j^D)a}\Pi(a)\varphi_j^D(x).$$

The condition $C\varphi = 0$ reads then

$$\sum_{j|\lambda+\lambda_j^D\in\sigma(A_0)} \alpha_j e^{-(\lambda+\lambda_j^D)a}\varphi_j^D|_{\mathcal{O}}(x) = 0, \qquad a\in(a_1,a_2).$$

Since the eigenfunctions of $-k\Delta$ with Dirichlet boundary conditions are analytic, we immediately obtain that $\varphi = 0$. ■

Main result

Theorem

Let $p_0 \in X$ *and assume that* $y(t) = p|_{(a_1,a_2)\times\mathcal{O}}$ *(*$t > 0$*) is known. Let* $\hat{p}$ *the observer defined by*

$$\begin{cases} \dot{\hat{p}}(t) = A\hat{p}(t) + L(C\hat{p}(t) - y(t)), & t \in (0,T) \\ \hat{p}(0) = 0, \end{cases}$$

where $L \in \mathcal{L}(Y,X)$ *is the stabilizing operator defined by*

$$L = i_{X_+} L_+ Q_+ \in \mathcal{L}(Y,X).$$

Then, there exist $M, \omega > 0$ *such that*

$$\|\hat{p}(t) - p(t)\| \leqslant Me^{-\omega t}\|p_0\|, \qquad t > 0.$$

Outline

Full observation in age

Taking $\Omega = (0, \pi)$ and assuming that p_0 is an unknown initial data, we want to estimate p at time $t = T$ where p solves:

$$\begin{cases} \partial_t p(a,x,t) + \partial_a p(a,x,t) & \\ \qquad = -\mu(a)p(a,x,t) + \partial_{xx}p(a,x,t), & a \in (0,a^*), x \in (0,\pi), t > 0, \\ p(a,0,t) = p(a,\pi,t) = 0, & a \in (0,a^*), t > 0, \\ p(a,x,0) = p_0(a,x), & a \in (0,a^*), x \in (0,\pi), \\ p(0,x,t) = \displaystyle\int_0^{a^*} \beta(a)p(a,x,t)\,\mathrm{d}a, & x \in (0,\pi), t > 0, \end{cases}$$

provided we know the observation

$$y(t) = p(t)_{|(0,a^*)\times(\pi/3,2\pi/3)}, \qquad t \in (0,T).$$

The fertility and mortality functions

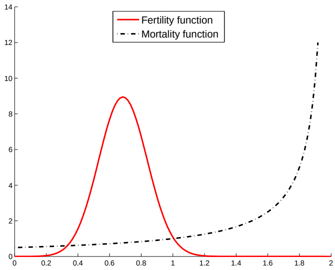


Figure: The fertility and mortality functions.

Taking $a^* = 2$, we choose the fertility and mortality function to be

$$\beta(a) = 10\,a(a^* - a)\exp\left\{-20(a - a^*/3)^2\right\}, \qquad \mu(a) = (a^* - a)^{-1}.$$

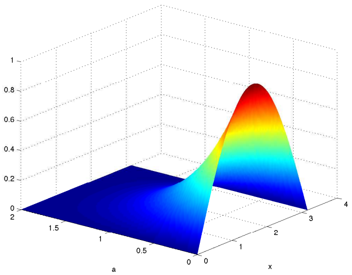
Note that the function $\Pi(a)$ can be computed explicitly:

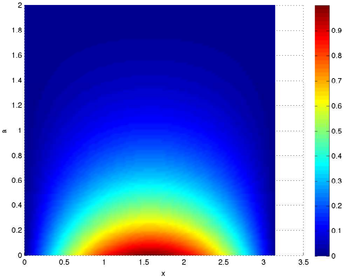
$$\Pi(a) = \exp\left(-\int_0^a \mu(s)\,\mathrm{d}s\right) = \frac{a^* - a}{a^*}.$$

First test : initial state = unstable eigenfunction

Under these assumptions, there is a unique unstable eigenvalue $\lambda_1 = \lambda_1^0 - \pi^2$(where $\lambda_1^0 \in \mathbb{R}$ satisfies $F(\lambda_1^0) = 1$). Computing numerically this value, we obtain that $\lambda_1 = 0.239$. We first choose as initial state an eigenfunction corresponding to λ_1

$$p_0(a,x) = \varphi_1(a,x) = \varphi_1^0(a)\varphi_1^D(x) = \frac{a^* - a}{a^*} e^{-\lambda_1^0 a} \sin(x).$$



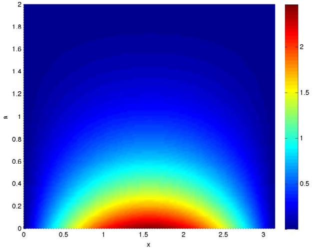


Estimated and exact solution at time $t = T$

The exact solution is:

$$p(a, x, t) = e^{\lambda_1 t} p_0(a, x).$$

We take: $T = 2a^*$, with $a^* = 2$.
Using $N_x = 100$, $N_a = 120$ and $N_t = 2N_a$, we obtain an L^2 relative error of 4.07%.

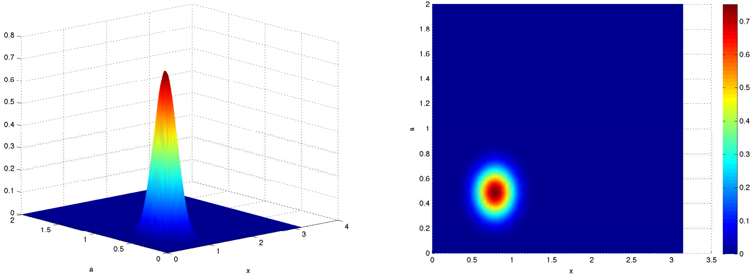


Estimated (left) and exact (right) solution at time $t = T$.

Second test : initial state = Gaussian function

We choose a space-aged localized initial distribution of population of gaussian type:

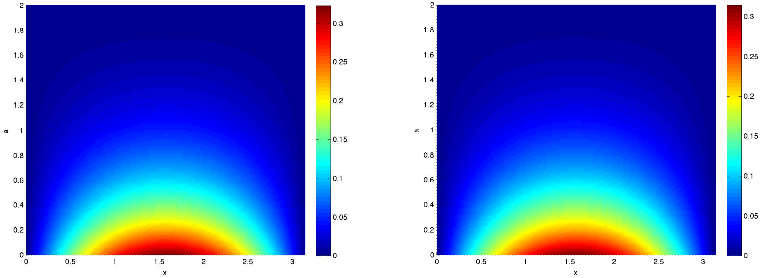
$$p_0(a,x) = \exp\left\{-\left(30(a-a^*/4)^2 + 20(x-\ell/4)^2\right)\right\}.$$



Gaussian initial state (3D and 2D representations).

Estimated and exact solution at time $t = T$

We obtain an L^2 relative error of 2.99%, 9.6% and 16.2% respectively for 5%, 10% and 15% of noise[1].

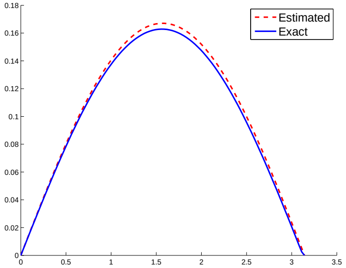


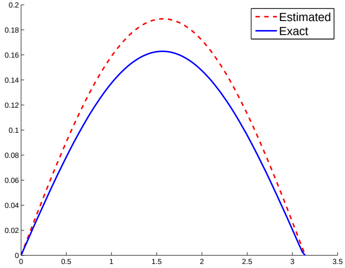
Estimated (left) and exact (right) solution at time $t = T$ (5% of noise).

[1] "Exact" solution refers here to a numerical solution computed numerically.

Estimated and exact final total population

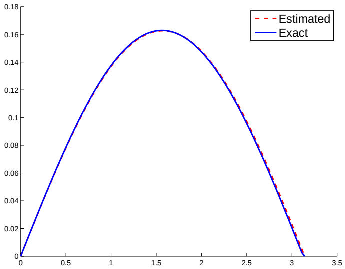
$$P_T(x) = \int_0^{a^*} p(a,T,x)\,\mathrm{d}a \qquad \text{and} \qquad \widehat{P}_T(x) = \int_0^{a^*} \widehat{p}(a,T,x)\,\mathrm{d}a.$$

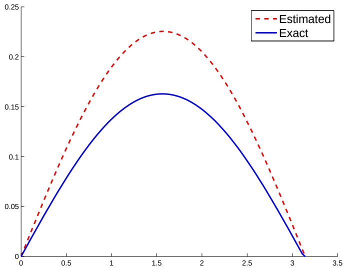




Estimated (dashed line) and exact total population at time $t = T$ with 5% of noise (left) and 15% of noise (right).

Distributed observation in space and age

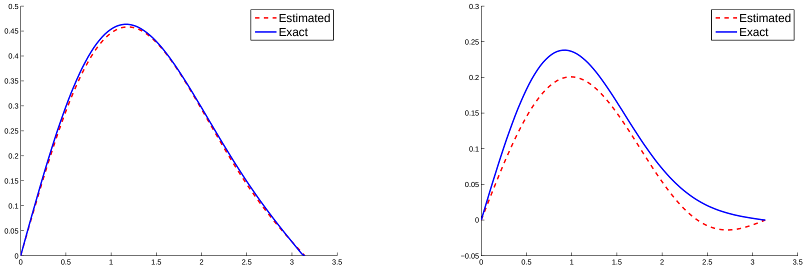




Estimated (dashed line) and exact total population at time $t = T$ with age observation in $\left(0, \dfrac{a^*}{20}\right)$ (left) and $\left(\dfrac{a^*}{2}, a^*\right)$ (right).

Influence of the observation time

We consider a configuration with two unstable eigenvalues and we investigate the influence of T. For $T = 0.5a^*$, we obtain a relative error of 27% for the population density, but we still obtain a reasonable approximation for the total population.



Estimated (dashed line) and exact total population at time $t = T$, for $T = a^*$ (left) and for $T = 0.5a^*$ (right).

Conclusion

1. **Other models**
   - Other outputs : $y(x,t) = \int_{a_1}^{a_2} p(a,x,t)\,\mathrm{d}a, \qquad x \in \mathcal{O}.$
   - Space dependent coefficients : $\beta(a,x), \qquad \mu(a,x).$
   - Nonlinearities: $\beta(a,x,P)$ and $\mu(a,x,P)$ where

$$P(x,t) := \int_0^{a^*} p(a,x,t)\,\mathrm{d}a.$$

   - adaptative observer which gives an estimation of p and k.
2. **Approximation**
   - Convergence analysis and error estimates
   - Uniform exponential stability (with respect to Δa et h)

Particular case: $A_+ := A|_{\mathcal{D}(A)\cap X_+}$ diagonalizable

The results collected here can be found in Barbu and Triggiani 2004. We assume that $A_+ := A|_{\mathcal{D}(A)\cap X_+}$ is diagonalizable. For simplicity, we denote by N the number of unstable eigenvalues of A counted with multiplicities (still denoted λ_k, $k = 1, \cdots, N$). This implies in particular that the unstable space is

$$X_+ = \bigoplus_{k=1}^{N} \text{Ker}\,(A - \lambda_k).$$

We denote then by $(\varphi_k)_{1\leqslant k\leqslant N}$ a basis of X_+. Denote by ψ_k an eigenfunction of A^* corresponding to the unstable eigenvalue $\overline{\lambda_k}$ $(1 \leqslant k \leqslant N)$. It can be shown that the family $(\psi_k)_{1\leqslant k\leqslant N}$ can be chosen such that $(\varphi_k)_{1\leqslant k\leqslant N}$ and $(\psi_k)_{1\leqslant k\leqslant N}$ form bi-orthogonal sequences, in the sense that $(\varphi_k, \psi_m)_X = \delta_{km}$. It follows then that the projection operator $P_+ \in \mathcal{L}(X, X_+)$ can be expressed as

$$P_+ z = \sum_{k=1}^{N} (z, \psi_k)_X\, \varphi_k \qquad (z \in X).$$

Since

$$X_+ = P_+ X = \text{Span}\{\varphi_k,\, 1 \leqslant k \leqslant N\},$$

it follows that

$$Y_+ = C X_+ = \text{Span}\{C\varphi_k,\, 1 \leqslant k \leqslant N\}.$$

Assume now that the family

$$(C\varphi_k)_{1\leqslant k\leqslant N} \text{ is linearly independent in } X. \qquad (1)$$

This property holds true in the case of internal observation. Therefore

$$\dim Y_+ = \dim X_+ = N.$$

We denote by $\mathbb{G}$ the Hermitian matrix of size $N \times N$ defined by

$$\mathbb{G} = \left((C\varphi_i, C\varphi_j)_Y\right)_{1\leqslant i\leqslant N,\, 1\leqslant j\leqslant N}.$$

It is not difficult to prove that (1) is equivalent to the fact that $\mathbb{G}$ is invertible.

The orthogonal projection operator Q_+

Lemma

Assume that property (1) *holds true. Then, for any* $y \in Y$, *then* $Q_+ y$ *is defined by*

$$Q_+ y = \sum_{i=1}^{N} (y, \eta_i)_Y \, C\varphi_i,$$

where

$$\eta_i = \sum_{j=1}^{N} \alpha_{ij} C\varphi_j$$

and

$$(\alpha_{ij})_{1 \leqslant i \leqslant N,\, 1 \leqslant j \leqslant N} = \mathbb{G}^{-1}.$$

From L_+ to L

The (finite dimensional) operator $C_+ \in \mathcal{L}(X_+, Y_+)$ satisfies $C_+\varphi_k = C\varphi_k$ for any $k \in \{1, ..., N\}$. Note that C_+ is nothing but the identity matrix when we choose as basis for X_+ and Y_+ respectively $(\varphi_k)_{1\leqslant k\leqslant N}$ and $(C\varphi_k)_{1\leqslant k\leqslant N}$. Therefore, using these bases, $A_+ + L_+C_+$ is a Hurwitz matrix provided $\mathrm{diag}(\lambda_1, ..., \lambda_N) + L_+$ is Hurwitz. It is thus sufficient to take $L_+ = -\sigma I$ with

$$\sigma > \mathrm{Re}\,\lambda_1$$

to ensure the stability of $A_+ + L_+C_+$.
The corresponding operator $L \in \mathcal{L}(Y, X)$ for every $y \in Y$

$$Ly = L_+Q_+y = L_+\left(\sum_{i=1}^{N} (y, \eta_i)_Y\, C\varphi_i\right) = -\sigma \sum_{i=1}^{N} (y, \eta_i)_Y\, \varphi_i,$$

and, following Theorem 6, $A + LC$ generates an exponentially stable semigroup.

Goal

Under these assumptions, there is a unique unstable eigenvalue $\lambda_1 = \lambda_1^0 - 1$ (where $\lambda_1^0 \in \mathbb{R}$ satisfies $F(\lambda_1^0) = 1$). Computing numerically this value, we obtain that $\lambda_1 = 0.239$.
The observation operator $C \in \mathcal{L}(X, Y)$ is given by

$$C\varphi = \varphi_{|(0,a^*)\times(\pi/3,2\pi/3)}, \qquad \forall \varphi \in X$$

where $X = L^2((0, a^*) \times (0, \pi))$ and $Y = L^2((0, a^*) \times (\pi/3, 2\pi/3))$.
In order to estimate $p(T)$, we use the observer designed previously.
As the unstable space is the one-dimensional space

$$X_+ = \text{Ker}\,(A - \lambda_1) = \text{Span}\,\{\varphi_1\} = \text{Span}\,\left\{\varphi_1^0(a)\varphi_1^D(x)\right\},$$

the observer involves the stabilizing output injection operator L defined by

$$Ly = -\sigma\,(y, \eta_1)_Y\,\varphi_1 \qquad (y \in Y),$$

where $\sigma > \lambda_1$ (gain coefficient) and

$$\eta_1 = \alpha_{11} C\varphi_1 = \frac{C\varphi_1}{\|C\varphi_1\|_Y^2}.$$

Goal

The observer solves then the following system

$$\begin{cases} \partial_t \widehat{p}(a,x,t) + \partial_a \widehat{p}(a,x,t) + \mu(a)\widehat{p}(a,x,t) & \\ \qquad -\partial_{xx}\widehat{p}(a,x,t) + \sigma\left(C\widehat{p},\eta_1\right)_Y \varphi_1(a,x) & \\ \qquad = \sigma\left(y,\eta_1\right)_Y \varphi_1(a,x), & a\in(0,a^*), x\in(0,\pi), t>0, \\ \widehat{p}(a,0,t) = \widehat{p}(a,\pi,t) = 0, & a\in(0,a^*), t>0, \\ \widehat{p}(a,x,0) = 0, & a\in(0,a^*), x\in(0,\pi), \\ \widehat{p}(0,x,t) = \displaystyle\int_0^{a^*} \beta(a)\widehat{p}(a,x,t)\,\mathrm{d}a, & x\in(0,\pi), t>0. \end{cases}$$

Goal: compare $p(T)$ and $\widehat{p}(T)$

Main difficulties concerning the discretization

- Singular behavior of the coefficient μ
  $\hookrightarrow$ rescaling the problem: introduce the auxiliary variable
  $u(a,x,t) = \frac{p(a,x,t)}{\Pi(a)} = \exp\left(\int_0^{a^*} \mu(s)\,\mathrm{d}s\right) p(a,x,t)$

$$\begin{cases} \partial_t u(a,x,t) + \partial_a u(a,x,t) - \partial_{xx} u(a,x,t) = 0, \\ u(a,0,t) = u(a,\pi,t) = 0, \\ u(a,x,0) = u_0(a,x) = p_0(a,x)/\Pi(a), \\ u(0,x,t) = \displaystyle\int_0^{a^*} m(a) u(a,x,t)\,\mathrm{d}a, \qquad \text{where } m(a) = \beta(a)\Pi(a) \end{cases}$$

- discretization of the renewal eq.: $p(0,x,t) = \int_0^{a^*} \beta(a) p(a,x,t)\,\mathrm{d}a$

$$\hookrightarrow u(0,x,n\Delta t) = \int_0^{a^*} m(a) u(a,x,(n-1)\Delta t)\,\mathrm{d}a$$

- presence of the extra term in the observer equation $(C\widehat{p},\eta_1)_Y\,\varphi_1$,
  $\hookrightarrow$ introduce $\theta(t) = (C\Pi\widehat{u},\eta_1)_Y$ which satisfies

$$\begin{cases} \dot{\theta}(t) = -(C\Pi\partial_a\widehat{u},\eta_1)_Y + (C\Pi\partial_{xx}\widehat{u},\eta_1)_Y - \sigma\theta(t) + \sigma\,(y,\eta_1)_Y \\ \theta(0) = 0 \end{cases}$$

Rescaling the open loop problem

First of all, in order to overcome the difficulties due to singular behavior of the coefficient μ, we introduce the auxiliary variable

$$u(a,x,t) = \frac{p(a,x,t)}{\Pi(a)} = \exp\left(\int_0^{a^*} \mu(s)\,\mathrm{d}s\right) p(a,x,t).$$

One can easily check that u satisfies

$$\begin{cases} \partial_t u(a,x,t) + \partial_a u(a,x,t) - \partial_{xx} u(a,x,t) = 0, & a \in (0,a^*), x \in (0,\pi), t > 0, \\ u(a,0,t) = u(a,\pi,t) = 0, & a \in (0,a^*),\ t > 0, \\ u(a,x,0) = u_0(a,x), & a \in (0,a^*),\ x \in (0,\pi), \\ u(0,x,t) = \displaystyle\int_0^{a^*} m(a) u(a,x,t)\,\mathrm{d}a, & x \in (0,\pi),\ t > 0, \end{cases}$$

where we have set $u_0(a,x) = p_0(a,x)/\Pi(a)$ and where $m(a) = \beta(a)\Pi(a)$ stands for the maternity function.

Finite difference discretization in time

Let $u^n(a,x)$ be an approximation of $u(a,x,t^n)$, where $t^n = n\Delta t$, $0 \leqslant n \leqslant N_t$, $\Delta t = T/N_t$ is a discretization of $(0,T)$. Starting from $u^0(a,x) = u_0(a,x)$, we construct u^n for $n \geqslant 1$ using an Euler's backwards scheme

$$
\begin{cases}
\dfrac{u^n(a,x) - u^{n-1}(a,x)}{\Delta t} + \partial_a u^n(a,x) \\
\qquad\qquad -\partial_{xx} u^n(a,x) = 0, & a \in (0,a^*),\ x \in (0,\pi), \\
u^n(a,0) = u^n(a,\pi) = 0, & a \in (0,a^*), \\
u^0(a,x) = u_0(a,x), & a \in (0,a^*),\ x \in (0,\pi), \\
u^n(0,x) = \displaystyle\int_0^{a^*} m(a) u^{n-1}(a,x)\,\mathrm{d}a, & x \in (0,\pi).
\end{cases}
$$

Finite difference discretization in space

Denoting by $u_i^n(a)$ an approximation of $u^n(x_i, a)$ (where $x_i = ih = i\ell/(N_x+1)$, with $0 \leqslant i \leqslant N_x+1$) and using a classical centered approximation for the second order derivative in space, the above system yields

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{U}^n}{\mathrm{d}a}(a) + \dfrac{1}{h^2}\mathbb{K}\mathbf{U}^n(a) + \dfrac{1}{\Delta t}\mathbf{U}^n(a) = \dfrac{1}{\Delta t}\mathbf{U}^{n-1}(a), \\ \mathbf{U}^n(0) = \displaystyle\int_0^{a^*} m(a)\mathbf{U}^{n-1}(a)\,\mathrm{d}a, \\ \mathbf{U}^0(a) = \mathbf{U}_0(a), \end{cases}$$

where

$$\mathbf{U}^n(a) = \begin{pmatrix} u_1^n(a) \\ \vdots \\ \vdots \\ u_{N_x}^n(a) \end{pmatrix}, \mathbf{U}_0(a) = \begin{pmatrix} u_0(a, x_1) \\ \vdots \\ \vdots \\ u_0(a, x_{N_x}) \end{pmatrix}, \mathbb{K} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & 0 \\ & \ddots & \ddots & \ddots \\ 0 & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}.$$

Finite difference discretization in age

We use a Crank-Nicholson scheme. Denoting by $u_i^{n,k}$ an approximation of $u_i^n(a^k)$, where $a^k = k\Delta a$, $0 \leqslant k \leqslant N_a$, $\Delta t = a^*/N_a$, and by

$$\mathbf{U}^{n,k} := \begin{pmatrix} u_1^{n,k} \\ \vdots \\ u_{N_x}^{n,k} \end{pmatrix}$$

an approximation of $\mathbf{U}^n(a^k)$, we move from age a^{k-1} to age a^k following

$$\frac{1}{\Delta a}\left(\mathbf{U}^{n,k} - \mathbf{U}^{n,k-1}\right) + \frac{1}{h^2}\mathbb{K}\left(\frac{\mathbf{U}^{n,k} + \mathbf{U}^{n,k-1}}{2}\right) + \frac{1}{\Delta t}\left(\frac{\mathbf{U}^{n,k} + \mathbf{U}^{n,k-1}}{2}\right) = \frac{1}{\Delta t}\left(\frac{\mathbf{U}^{n-1,k} + \mathbf{U}^{n-1,k-1}}{2}\right),$$

with the initial conditions

$$\begin{cases} \mathbf{U}^{0,k} = \mathbf{U}_0(a^k), \qquad \forall k = 0, \ldots, N_a, \\ \mathbf{U}^{n,0} = \displaystyle\sum_{k=0}^{N_a} \omega_k \, m(a^k)\mathbf{U}^{n-1,k} \simeq \int_0^{a^*} m(a)\mathbf{U}^{n-1}(a)\,\mathrm{d}a. \end{cases}$$

The algorithm

1. For $n = 0$: Initialization of $\mathbf{U}^{0,k}$
2. For $n = 1, \dots, N_t$:
   - $k = 0$: Initialization of $\mathbf{U}^{n,0}$ using the values of $(\mathbf{U}^{n-1,j})_{j=0}^{N_a}$:
     $$\mathbf{U}^{n,0} = \sum_{k=0}^{N_a} \omega_k \, m(a^k) \mathbf{U}^{n-1,k}$$
   - For $k = 1, \dots, N_a$, $\mathbf{U}^{n,k} = \begin{pmatrix} u_1^{n,k} \\ \vdots \\ u_{N_x}^{n,k} \end{pmatrix}$ solves the linear system
     $$\mathbb{A}\mathbf{U}^{n,k} = \mathbf{b}^{n,k}$$
     where
     $$\mathbb{A} = \left(\Delta t + \frac{1}{2}\Delta a\right) \mathbb{I} + \frac{1}{2}\frac{\Delta t \, \Delta a}{h^2} \mathbb{K}$$
     $$\mathbf{b}^{n,k} = \frac{\Delta a}{2}\left(\mathbf{U}^{n-1,k} + \mathbf{U}^{n-1,k-1}\right) + \left[\left(\Delta t - \frac{\Delta a}{2}\right)\mathbb{I} - \frac{\Delta t \, \Delta a}{2h^2}\mathbb{K}\right]\mathbf{U}^{n,k-1}$$
   - $\mathbf{Y}^{n,k} = \begin{pmatrix} y_1^{n,k} \\ \vdots \\ y_{N}^{n,k} \end{pmatrix}$ where $y_i^{n,k} = \Pi(a^k) u_i^{n,k}$ if $\ell_1 \leqslant ih \leqslant \ell_2$ and

Discretization of the closed loop system : observer design

$$
\begin{cases}
\partial_t \widehat{p}(a,x,t) + \partial_a \widehat{p}(a,x,t) + \mu(a)\widehat{p}(a,x,t) \\
\qquad -\partial_{xx}\widehat{p}(a,x,t) + \sigma\left(C\widehat{p},\eta_1\right)_Y \varphi_1(a,x) = \sigma\left(y,\eta_1\right)_Y \varphi_1(a,x), \\
\widehat{p}(a,0,t) = \widehat{p}(a,\pi,t) = 0, \\
\widehat{p}(a,x,0) = 0, \\
\widehat{p}(0,x,t) = \int_0^{a^*} \beta(a)\widehat{p}(a,x,t)\,\mathrm{d}a.
\end{cases}
$$

Rescaling the problem

First of all, we introduce the auxiliary variable

$$\widehat{u}(a,x,t)=\frac{\widehat{p}(a,x,t)}{\Pi(a)}=\exp\left(\int_0^{a^*}\mu(s)\,\mathrm{d}s\right)\widehat{p}(a,x,t).$$

One can easily check that $\widehat{u}$ satisfies

$$\begin{cases}\partial_t\widehat{u}(a,x,t)+\partial_a\widehat{u}(a,x,t)-\partial_{xx}\widehat{u}(a,x,t)\\ \qquad\qquad +\sigma\left(C\Pi\widehat{u},\eta_1\right)_Y v_1(a,x)=\sigma\left(y,\eta_1\right)_Y v_1(a,x),\\ \widehat{u}(a,0,t)=\widehat{u}(a,\pi,t)=0,\\ \widehat{u}(a,x,0)=0,\\ \widehat{u}(0,x,t)=\displaystyle\int_0^{a^*} m(a)\widehat{u}(a,x,t)\,\mathrm{d}a,\end{cases}$$

where we have set $v_1(a,x)=\varphi_1(a,x)/\Pi(a)$.

Discretization of the term $(C\Pi\widehat{u}, \eta_1)_Y$

Let us introduce

$$\theta(t) = (C\Pi\widehat{u}, \eta_1)_Y \,.$$

Using the fact that $(C\Pi v_1, \eta_1)_Y = 1$, we remark that θ satisfies

$$\dot{\theta}(t) = -\,(C\Pi\partial_a\widehat{u}, \eta_1)_Y + (C\Pi\partial_{xx}\widehat{u}, \eta_1)_Y - \sigma\theta(t) + \sigma\,(y, \eta_1)_Y \,.$$

Consequently,

$$\begin{cases} \dot{\theta}(t) = -\,(C\Pi\partial_a\widehat{u}(t), \eta_1)_Y + (C\Pi\partial_{xx}\widehat{u}(t), \eta_1)_Y - \sigma\theta(t) + \sigma\,(y(t), \eta_1)_Y \,, \\ \partial_t\widehat{u}(a,x,t) + \partial_a\widehat{u}(a,x,t) - \partial_{xx}\widehat{u}(a,x,t) + \sigma\theta(t)v_1(a,x) \\ \qquad\qquad = \sigma\,(y, \eta_1)_Y\, v_1(a,x), \\ \theta(0) = 0, \\ \widehat{u}(a,0,t) = \widehat{u}(a,\ell,t) = 0, \\ \widehat{u}(a,x,0) = 0, \\ \widehat{u}(0,x,t) = \displaystyle\int_0^{a^*} m(a)\widehat{u}(a,x,t)\,\mathrm{d}a. \end{cases}$$

Finite difference discretization in time

Let $\widehat{u}^n(a,x)$ (resp. θ^n, $y^n(a,x)$) be an approximation of $\widehat{u}(a,x,t^n)$ (resp. $\theta(t^n)$, $y(a,x,t^n)$), where $t^n = n\Delta t$, $0 \leqslant n \leqslant N_t$, $\Delta t = T/N_t$ is a discretization of $(0,T)$. Starting from $\theta^0 = 0$ and $\widehat{u}^0(a,x) = 0$, we construct θ^n and $\widehat{u}^n$ for $n \geqslant 1$ using an Euler's backwards scheme

$$\begin{cases} \dfrac{1}{\Delta t}\left(\theta^n - \theta^{n-1}\right) = -\left(C\Pi\partial_a \widehat{u}^{n-1}, \eta_1\right)_Y + \left(C\Pi\partial_{xx}\widehat{u}^{n-1}, \eta_1\right)_Y \\ \qquad\qquad -\sigma\theta^n + \sigma\left(y^n, \eta_1\right)_Y, \\ \dfrac{1}{\Delta t}\left(\widehat{u}^n(a,x) - \widehat{u}^{n-1}(a,x)\right) + \partial_a\widehat{u}^n(a,x) - \partial_{xx}\widehat{u}^n(a,x) + \sigma\theta^n v_1 \\ \qquad\qquad = \sigma\left(y^n, \eta_1\right)_Y v_1, \\ \theta^0 = 0, \\ \widehat{u}^n(a,0) = \widehat{u}^n(a,\ell) = 0, \\ \widehat{u}^0(a,x) = 0, \\ \widehat{u}^n(0,x) = \displaystyle\int_0^{a^*} m(a)\widehat{u}^{n-1}(a,x)\,\mathrm{d}a. \end{cases}$$

Finite difference discretization in space

Denoting by $\widehat{u}_i^n(a)$ an approximation of $\widehat{u}^n(x_i, a)$, the above system yields

$$
\begin{cases}
\dfrac{1}{\Delta t}\theta^n = \dfrac{1}{\Delta t}\theta^{n-1} - \sigma\theta^n - h\left(\displaystyle\int_0^{a^*} \Pi(a)\boldsymbol{\eta}_1(a)^{\mathbf{T}}\partial_a\widehat{\mathbf{U}}^{n-1}(a)\,\mathrm{d}a\right) \\
\qquad -\dfrac{1}{h}\left(\displaystyle\int_0^{a^*} \Pi(a)\boldsymbol{\eta}_1(a)^{\mathbf{T}}\mathbb{K}\widehat{\mathbf{U}}^{n-1}(a)\,\mathrm{d}a\right) + \sigma h\left(\displaystyle\int_0^{a^*} \boldsymbol{\eta}_1(a)^{\mathbf{T}}\mathbf{Y}^n(a)\,\mathrm{d}a\right), \\
\dfrac{\mathrm{d}\widehat{\mathbf{U}}^n}{\mathrm{d}a}(a) + \dfrac{1}{h^2}\mathbb{K}\widehat{\mathbf{U}}^n(a) + \dfrac{1}{\Delta t}\widehat{\mathbf{U}}^n(a) + \sigma\theta^n\mathbf{V}_1(a) \\
\quad = \dfrac{1}{\Delta t}\widehat{\mathbf{U}}^{n-1}(a) + \sigma h\left(\displaystyle\int_0^{a^*} \boldsymbol{\eta}_1(a)^{\mathbf{T}}\mathbf{Y}^n(a)\,\mathrm{d}a\right)\mathbf{V}_1(a), \\
\theta^0 = 0, \\
\widehat{\mathbf{U}}^n(0) = \displaystyle\int_0^{a^*} m(a)\widehat{\mathbf{U}}^{n-1}(a)\,\mathrm{d}a, \\
\widehat{\mathbf{U}}^0(a) = 0,
\end{cases}
$$

where

$$
\widehat{\mathbf{U}}^n(a) = \begin{pmatrix} \widehat{u}_1^n(a) \\ \vdots \\ \widehat{u}_{N_x}^n(a) \end{pmatrix}, \mathbf{V}_1(a) = \begin{pmatrix} v_1(a, x_1) \\ \vdots \\ v_1(a, x_{N_x}) \end{pmatrix}, \boldsymbol{\eta}_1(a) = \begin{pmatrix} \eta_1(a, x_1) \\ \vdots \\ \eta_1(a, x_{N_x}) \end{pmatrix}.
$$

Finite difference discretization in age

We use a Crank-Nicholson scheme. Denoting by $\widehat{u}_i^{n,k}$ an approximation of $\widehat{u}_i^n(a^k)$, where $a^k = k\Delta a$, $0 \leqslant k \leqslant N_a$, $\Delta t = a^*/N_a$, and by $\widehat{\mathbf{U}}^{n,k} := \begin{pmatrix} \widehat{u}_1^{n,k} \\ \vdots \\ \widehat{u}_{N_x}^{n,k} \end{pmatrix}$ an approximation of $\widehat{\mathbf{U}}^n(a^k)$, we move from age a^{k-1} to age a^k following

$$\frac{1}{\Delta t}\theta^n = \frac{1}{\Delta t}\theta^{n-1} - \sigma\theta^n - h\Delta a\left(\sum_{k=1}^{N_a}\Pi(k\Delta a)\left(\boldsymbol{\eta}_1^k\right)^{\mathbf{T}}\left(\frac{\widehat{\mathbf{U}}^{n-1,k}-\widehat{\mathbf{U}}^{n-1,k-1}}{\Delta a}\right)\right)$$

$$-\frac{\Delta a}{h}\left(\sum_{k=1}^{N_a}\Pi(k\Delta a)\left(\boldsymbol{\eta}_1^k\right)^{\mathbf{T}}\mathbb{K}\widehat{\mathbf{U}}^{n-1,k}\right) + \sigma h\Delta a\left(\sum_{k=1}^{N_a}\left(\boldsymbol{\eta}_1^k\right)^{\mathbf{T}}\mathbf{Y}^{n,k}\right),$$

and

$$\frac{1}{\Delta a}\left(\widehat{\mathbf{U}}^{n,k}-\widehat{\mathbf{U}}^{n,k-1}\right) + \frac{1}{h^2}\mathbb{K}\left(\frac{\widehat{\mathbf{U}}^{n,k}+\widehat{\mathbf{U}}^{n,k-1}}{2}\right) + \frac{1}{\Delta t}\left(\frac{\widehat{\mathbf{U}}^{n,k}+\widehat{\mathbf{U}}^{n,k-1}}{2}\right)$$

$$+\sigma\theta^n\mathbf{V}_1^k = \frac{1}{\Delta t}\left(\frac{\widehat{\mathbf{U}}^{n-1,k}+\widehat{\mathbf{U}}^{n-1,k-1}}{2}\right) + \sigma h\Delta a\left(\sum_{j=1}^{N_a}\left(\boldsymbol{\eta}_1^j\right)^{\mathbf{T}}\mathbf{Y}^{n,j}\right)\mathbf{V}_1^k,$$

with the initial conditions

$$\begin{cases} \theta^0 = 0, \\ \widehat{\mathbf{U}}^{0,k} = 0, \qquad \forall k = 0, \dots, N_a, \\ \widehat{\mathbf{U}}^{n,0} = \sum_{k=0}^{N_a} \omega_k \, m(a^k) \widehat{\mathbf{U}}^{n-1,k}. \end{cases}$$

Here

$$\mathbf{V}_1^k = \begin{pmatrix} v_1(k\Delta a, x_1) \\ \vdots \\ v_1(k\Delta a, x_{N_x}) \end{pmatrix}, \boldsymbol{\eta}_1^k = \begin{pmatrix} \eta_1(k\Delta a, x_1) \\ \vdots \\ \eta_1(k\Delta a, x_{N_x}) \end{pmatrix}, \mathbf{Y}^{n,k} = \begin{pmatrix} y^n(k\Delta a, x_1) \\ \vdots \\ y^n(k\Delta a, x_{N_x}) \end{pmatrix}.$$

1. For $n = 0$: Initialization of θ^0 and $\widehat{\mathbf{U}}^{0,k}$ $(k = 0, \ldots, N_a)$ at 0.
2. For $n = 1, \ldots, N_t$:
   - Calculate θ^n using the values of θ^{n-1} and $(\widehat{\mathbf{U}}^{n-1,j})_{j=0}^{N_a}$:

$$\theta^n = \frac{1}{1+\sigma\Delta t}\left(\theta^{n-1} - h\Delta t \sum_{k=1}^{N_a} \Pi(k\Delta a)(\boldsymbol{\eta}_1^k)^{\mathbf{T}}\left(\widehat{\mathbf{U}}^{n-1,k} - \widehat{\mathbf{U}}^{n-1,k-1}\right)\right.$$
$$\left. - \frac{\Delta a \Delta t}{h}\sum_{k=1}^{N_a} \Pi(k\Delta a)(\boldsymbol{\eta}_1^k)^{\mathbf{T}} \mathbb{K} \widehat{\mathbf{U}}^{n-1,k} + \sigma h \Delta a \Delta t \sum_{k=1}^{N_a} (\boldsymbol{\eta}_1^k)^{\mathbf{T}} \mathbf{Y}^{n,k}\right)$$

   - $k = 0$: Initialization of $\widehat{\mathbf{U}}^{n,0}$ using the values of $(\widehat{\mathbf{U}}^{n-1,j})_{j=0}^{N_a}$:

$$\widehat{\mathbf{U}}^{n,0} = \sum_{k=0}^{N_a} \omega_k \, m(k\Delta a) \widehat{\mathbf{U}}^{n-1,k}$$

   - For $k = 1, \ldots, N_a$, solve the linear system

$$\mathbb{A}\widehat{\mathbf{U}}^{n,k} = \widehat{\mathbf{b}}^{n,k}$$

where

$$\widehat{\mathbf{b}}^{n,k} = \frac{\Delta a}{2}\left(\widehat{\mathbf{U}}^{n-1,k} + \widehat{\mathbf{U}}^{n-1,k-1}\right) + \left[(\Delta t - \frac{\Delta a}{2})\mathbb{I} - \frac{\Delta t \, \Delta a}{2h^2}\mathbb{K}\right]\widehat{\mathbf{U}}^{n,k-1}$$
$$- \sigma \Delta a \Delta t \, \theta^n \mathbf{V}_1^k + \sigma h (\Delta a)^2 \Delta t \left(\sum_{j=1}^{N_a} \left(\boldsymbol{\eta}_1^j\right)^{\mathbf{T}} \mathbf{Y}^{n,j}\right) \mathbf{V}_1^k$$

End of the algorithm

- $\widehat{\mathbf{P}^{n,k}} = \begin{pmatrix} \widehat{p}_1^{n,k} \\ \vdots \\ \widehat{p}_{N_x}^{n,k} \end{pmatrix}$ where $\widehat{p}_i^{n,k} = \Pi(a^k)\widehat{u}_i^{n,k}$.