Exact controllability of a system of coupled wave equations with only one boundary control

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   - Observability Inequalies
   - Exact Controllability

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Previous results

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2, such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. In a previous work, Toufayli and Wehbe considered the energy decay rate of a multidimensional system of wave equations coupled by velocities:

$$\begin{aligned}
u_{tt} - \Delta u + b y_t &= 0, && \text{in } \Omega \times \mathbb{R}_+, && (1.1)\\
y_{tt} - a\Delta y - b u_t &= 0, && \text{in } \Omega \times \mathbb{R}_+, && (1.2)\\
u &= 0, && \text{on } \Gamma \times \mathbb{R}_+, && (1.3)\\
y &= 0, && \text{on } \Gamma_0 \times \mathbb{R}_+, && (1.4)\\
\partial_\nu y + y_t &= 0, && \text{on } \Gamma_1 \times \mathbb{R}_+ && (1.5)
\end{aligned}$$

with the following initial data

$$(u(x,0), y(x,0)) = (u_0, y_0),\ (u_t(x,0), y_t(x,0)) = (u_1, y_1) \qquad (1.6)$$

where $a > 0$ and $b \in \mathbb{R}$.

Previous results

- Under the equal speed wave propagation condition (in the case $a = 1$) and if the coupling parameter b is small enough, we established an exponential energy decay estimate. However, on the contrary, no stability type has been discussed.

Previous results

- Under the equal speed wave propagation condition (in the case $a = 1$) and if the coupling parameter b is small enough, we established an exponential energy decay estimate. However, on the contrary, no stability type has been discussed.
- Recently, Najdi and Wehbe, considered the same system in an one-dimensional domain. They established the following stability results
  - Strong if and only if

$$b^2 \neq \frac{(k_1^2 - a k_2^2)(a k_1^2 - k_2^2)\pi^2}{(a+1)(k_1^2 + k_2^2)}, \quad \forall k_1, k_2 \in \mathbb{Z}. \qquad \text{(SC1)},$$

  - Uniform iff (SC1) hods, $a = 1$ and $b \neq k\pi$, $\forall k \in \mathbb{Z}$,
  - Polynomial of type $\frac{1}{\sqrt{t}}$ if (SC1) hods, $a = 1$ and $b = k\pi$, $k \in \mathbb{Z}$,
  - Polynomial of type $\frac{1}{\sqrt{t}}$ if (SC1) hods, $a \in \mathbb{Q}$ and b small enough **or** $\sqrt{a} \in \mathbb{Q}$.

Objective

Our objective is to investigate the indirect exact boundary controllability of the following system

$$
\begin{aligned}
u_{tt} - \Delta u + b y_t &= 0, && \text{in } \Omega \times \mathbb{R}_+, && (1.7)\\
y_{tt} - a\Delta y - b u_t &= 0, && \text{in } \Omega \times \mathbb{R}_+, && (1.8)\\
u &= 0, && \text{on } \Gamma \times \mathbb{R}_+, && (1.9)\\
y &= 0 && \text{on } \Gamma_0 \times \mathbb{R}_+, && (1.10)\\
y &= v(t), && \text{on } \Gamma_1 \times \mathbb{R}_+ && (1.11)
\end{aligned}
$$

with the following initial data

$$(u(x,0), y(x,0)) = (u_0, y_0),\ (u_t(x,0), y_t(x,0)) = (u_1, y_1) \qquad (1.12)$$

Control is applied only to the second equation. The first equation is controlled indirectly by means of the coupling of the equations.

History

The boundary indirect exact controllability of a system of wave equations coupled through the zero order terms has been studied with different approaches. We recall the results of Alabau and Liu-Rao

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- Fatiha Alabau (in 2003), studied the indirect boundary observability of a system of wave equations coupled through the zero order terms with same speed of propagation. Using a multiplier method, she proved that, for sufficiently large time T, the observation of the trace of the normal derivative of the first component of the solution on Γ_1 allows us to get back a weakened energy of the initial data. Then the system is exactly controllable by means of a one boundary control.

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- Liu and Rao (in 2009), extended the result of Alabau by considering the important case when the waves propagate with different speeds within the two equations. Using a spectral approach, they studied how the modes of the uncontrolled equation are influenced by the modes of the controlled equation and to get the optimal right controllability spaces.

Homogeneous system

We consider the following homogeneous system

$$\varphi_{tt} - \Delta\varphi + b\psi_t = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{1.13}$$

$$\psi_{tt} - \Delta\psi - b\varphi_t = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{1.14}$$

$$\varphi = \psi = 0, \qquad \text{on } \Gamma \times \mathbb{R}_+, \tag{1.15}$$

with the following initial data

$$\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x),$$

$$\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \; x \in \Omega, \tag{1.16}$$

Well-posedness of homogeneous system

Let (φ, ψ) Be a regular solution of system (1.13)-(1.16), we define

$$E_H(t) = \frac{1}{2}\int_\Omega (|\varphi_t|^2 + |\nabla\varphi|^2 + |\psi_t|^2 + |\nabla\psi|^2)dx. \tag{1.17}$$

It is easy to see that $E_H'(t) = 0$, then system (1.13)-(1.16) is conservative in the sens that its energy is constant. Now, we define the Hilbert space

$$\mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2 \tag{1.18}$$

such that, for all $\Phi = (\varphi, \xi, \psi, \varrho)$, $\widetilde{\Phi} = (\widetilde{\varphi}, \widetilde{\xi}, \widetilde{\psi}, \widetilde{\varrho})$ in $\mathcal{H}$, we have

$$(\Phi, \widetilde{\Phi})_{\mathcal{H}} := \int_\Omega \left(\nabla\varphi \cdot \nabla\widetilde{\varphi} + \xi\widetilde{\xi} + \nabla\psi \cdot \nabla\widetilde{\psi} + \varrho\widetilde{\varrho}\right)dx. \tag{1.19}$$

Well-posedness of homogeneous system

We define the unbounded linear operator $\mathcal{A}$ by :

$$D(\mathcal{A}) = \{\Phi = (\varphi, \xi, \psi, \varrho) \in \mathcal{H} : \varphi,\ \psi\ \in H^2(\Omega) \cap H_0^1(\Omega),\ \xi, \varrho \in H_0^1(\Omega)\},$$

$$\mathcal{A}\Phi = (\xi, \Delta\varphi - b\varrho, \varrho, \Delta\psi + b\xi),\ \ \forall\, \Phi = (\varphi, \xi, \psi, \varrho) \in D(\mathcal{A}).$$

Then system (1.13)-(1.16) is equivalent to

$$\Phi_t = \mathcal{A}\Phi,\ \ \Phi(0) = \Phi_0 \in \mathcal{H}. \tag{1.20}$$

By semi-group theory, system (1.20) admits unique solution Φ such that

$$\Phi(t) \in C^0(0, +\infty; \mathcal{H}),\ \text{if}\ \Phi_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1) \in \mathcal{H},$$

$$\Phi(t) \in C^0(0, +\infty; D(\mathcal{A})) \cap C^1(0, +\infty; \mathcal{H}),\ \text{if}\ \Phi_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1) \in D(\mathcal{A})$$

Observability Inequalities

Assume that there exists $\delta > 0$ and $x_0 \in \mathbb{R}^N$ such that, putting $m(x) = x - x_0$, we have

$$(m \cdot \nu) \geq \delta^{-1}, \;\; \forall x \in \Gamma_1 \;\text{ and }\; (m \cdot \nu) \leq 0, \;\; \forall x \in \Gamma_0. \tag{GC}$$

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Theorem

Assume that (GC) *holds , a=1 and* $0 < b < b_0 = \frac{1}{4R+3\max\{1,c_0\}}$, *where* c_0 *is the Poincaré constant. Then, there exists* $T_0 > 0$, *such that for all* $T > T_0$ *and for all* $\Phi_0 \in \mathcal{H}$,*the weak solution* Φ *of (1.20) verifying*

$$c_2 \int_0^T \int_{\Gamma_1} |\partial_\nu \psi|^2 d\Gamma dt \leq \|\Phi_0\|_{\mathcal{H}}^2 \leq c_1 \int_0^T \int_{\Gamma_1} |\partial_\nu \psi|^2 d\Gamma dt \tag{1.21}$$

where c_1, c_2 *are positive constants and*

$$T_0 = \frac{\frac{6}{b} + 8R + 6\max\{1, c_0\}}{1 - b(4R + 3\max\{1, c_0\})}.$$

Sketch of the proof

- Multiply equation (1.13) by ψ_t and (1.14) by φ_t respectively, we get

$$b\int_0^T\int_\Omega |\varphi_t|^2 dxdt \leq b\int_0^T\int_\Omega |\psi_t|^2 dxdt + \|\Phi_0\|^2 . \qquad (1.22)$$

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- Multiply equation (1.14) by $(N-1)\psi + 2(m\cdot\nabla\psi)$, we get

$$\begin{aligned} 2\int_0^T\int_\Omega|\psi_t|^2dxdt + 2\int_0^T\int_\Omega|\nabla\psi|^2dxdt \\ -2\int_0^T\int_{\Gamma_0}(m\cdot\nu)|\partial_\nu\psi|^2d\Gamma dt \\ -2\int_0^T\int_{\Gamma_1}(m\cdot\nu)|\partial_\nu\psi|^2d\Gamma dt \\ \leq C\|\Phi_0\|^2 + \tilde{C}bT\|\Phi_0\|^2. \end{aligned} \qquad (1.23)$$

Sketch of the proof

- Multiply equation (1.13) by φ_t, we get

$$-\int_0^T \int_\Omega |\varphi_t|^2 dxdt + \int_0^T \int_\Omega |\nabla\varphi|^2 dxdt \leq C \|\Phi_0\|^2 + \tilde{C} bT \|\Phi_0\|^2 . \tag{1.24}$$

Sketch of the proof

- Multiply equation (1.13) by φ_t, we get

$$-\int_0^T \int_\Omega |\varphi_t|^2 dxdt + \int_0^T \int_\Omega |\nabla\varphi|^2 dxdt \leq C\,\|\Phi_0\|^2 + \tilde{C}bT\,\|\Phi_0\|^2 . \tag{1.24}$$

- Combining equations (1.22), (1.25) and (1.24) and use the geometric condition (GC), we get

$$T\,\|\Phi_0\|^2 = \int_0^T \|\Phi(t)\|^2\,dt - \int_0^T \int_{\Gamma_1} (m\cdot\nu)|\partial_\nu\psi|^2 d\Gamma dt \leq C\,\|\Phi_0\|^2 + \tilde{C}bT\,\|\Phi_0\|^2 . \tag{1.25}$$

Sketch of the proof

- Multiply equation (1.13) by φ_t, we get

$$-\int_0^T \int_\Omega |\varphi_t|^2 dxdt + \int_0^T \int_\Omega |\nabla \varphi|^2 dxdt \leq C \left\|\Phi_0\right\|^2 + \tilde{C} b T \left\|\Phi_0\right\|^2 . \tag{1.24}$$

- Combining equations (1.22), (1.25) and (1.24) and use the geometric condition (GC), we get

$$T \left\|\Phi_0\right\|^2 = \int_0^T \left\|\Phi(t)\right\|^2 dt - \int_0^T \int_{\Gamma_1} (m \cdot \nu) |\partial_\nu \psi|^2 d\Gamma dt \leq C \left\|\Phi_0\right\|^2 + \tilde{C} b T \left\|\Phi_0\right\|^2 . \tag{1.25}$$

Choosing $b \leq \dfrac{1}{\tilde{C}}$ we deduce the inverse observability inequality.

Exact Controllability result

Theorem

Let $T > 0$ *and* $v \in L^2(]0, T[, L^2(\Gamma_1))$. *For all initial data* $U_0 = (u_0, u_1, y_0, y_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^2$, *the system (1.5) admits a unique weak solution*

$$U(x, t) \in C^0([0, T], (L^2(\Omega) \times H^{-1}(\Omega))^2).$$

In addition, we have the continuous linear mapping

$$(U_0, v) \longrightarrow (U, U_t). \tag{1.26}$$

Controlled system

Theorem

Assume that $0 < b < b_0$. *For all* $T > T_0$ *où* b_0, T_0 *and for all*

$$U_0 \in (L^2(\Omega) \times H^{-1}(\Omega))^2,$$

there exists a control $v(t) \in L^2(0, T, L^2(\Gamma_1))$ *such that the solution* $U = (u, u_t, y, y_t)$ *of the controlled system (1.5) satisfies* $u(T) = u_t(T) = y(T) = y_t(T) = 0$.

Sketch of the proof

Thanks to observability inequalities (1.21), we deduce that

$$\| \Phi_0 \|_{\mathcal{H}}^2 = \int_0^T \int_{\Gamma_1} |\frac{\partial \psi}{\partial \nu}|^2 d\Gamma dt, \tag{1.27}$$

is a norm. Choosing $v = -\dfrac{\partial \psi}{\partial \nu} \in L^2(0, T, L^2(\Gamma))$.
and solve the following problem

$$\begin{cases} \chi_{tt} - \Delta \chi + b\zeta_t = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \zeta_{tt} - \Delta \zeta - b\chi_t = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \chi = 0, & \text{on } \Gamma \times \mathbb{R}^+, \\ \zeta = 0, & \text{on } \Gamma_0 \times \mathbb{R}^+, \\ \zeta = -\frac{\partial \psi}{\partial \nu}, & \text{on } \Gamma_1 \times \mathbb{R}^+, \\ \chi(T) = \chi_t(T) = \zeta(T) = \zeta_t(T) = 0, & \text{in } \Omega. \end{cases} \tag{1.28}$$

Sketch of the proof

We define the linear operator $\Lambda : \mathcal{H} \longrightarrow (H^{-1}(\Omega) \times L^2(\Omega))^2$

$$\Lambda\Phi_0 = (\chi_t(0), -\chi(0), \zeta_t(0), -\zeta(0)), \qquad \forall \Phi_0 \in \mathcal{H}. \tag{1.29}$$

Thanks to the inverse observability inequality we deduce that Λ is an isomorphism. In particular, for all
$U_0 = (u_1, -u_0, y_1, -y_0) \in \ (L^2(\Omega) \times H^{-1}(\Omega))^2$, there exists $\Phi_0 \in \mathcal{H}$, such that

$$\Lambda\Phi_0 = (u_1, -u_0, y_1, -y_0).$$

Then

$$(u, u_t, y, y_t) = (\chi, \chi_t, \zeta, \zeta_t).$$

Consequently

$$u(T) = u_t(T) = y(T) = y_t(T) = 0.$$

Question

What happens in the One-dimensional case??

Image

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Controlled system

The aim of this part is to investigate the exact boundary controllability of the following one-dimensional system:

$$\begin{cases} u_{tt} - u'' + by_t = 0, \\ y_{tt} - ay'' - bu_t = 0, \\ u(1,t) = u(0,t) = y(0,t), \\ y(1,t) = v(t) \end{cases} \tag{2.1}$$

where $a > 0$, $b \in \mathbb{R}$ are constants and v is the control applied to the second equation at the right boundary. We start by considering the homogeneous system in the case $a = 1$.

Homogeneous system

Let us consider the following homogeneous system

$$\begin{cases} \varphi_{tt} - \varphi'' + b\psi_t = 0, \\ \psi_{tt} - \psi'' - b\varphi_t = 0, \\ \varphi(0,t) = \varphi(1,t) = 0, \\ \psi(0,t) = \psi(1,t) = 0. \end{cases} \tag{2.2}$$

Let us define the energy space $\mathcal{H} = (H_0^1(0,1) \times L^2(0,1))^2$ such that, for all $\Phi = (\varphi, \omega, \psi, \eta)$, $\widetilde{\Phi} = (\widetilde{\varphi}, \widetilde{\omega}, \widetilde{\psi}, \widetilde{\eta})$, we have

$$\left(\Phi, \widetilde{\Phi}\right)_{\mathcal{H}} = \int \left(\varphi'\widetilde{\varphi}' + \omega\widetilde{\omega} + \psi'\widetilde{\psi}' + \eta\widetilde{\eta}\right) dx.$$

We define the linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \longrightarrow \mathcal{H}$ by

$$D(\mathcal{A}) = (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1))^2$$

$$\mathcal{A} = (\varphi, \omega, \psi, \eta) = (\omega, \varphi'' - b\eta, \eta, \psi'' + b\omega).$$

Observability inequality

We will establish the following observability result

Theorem

Assume that $a = 1$, there exist no $k \in \mathbb{Z}$ such that $b = k\pi$ and

$$T > \frac{2\pi}{\pi + |b|}. \tag{2.3}$$

Then, there exists a constant $c > 0$ depending only on b, such that the following inverse observability inequality holds

$$c \left\|(\phi_0, \phi_1, \psi_0, \psi_1)\right\|_{\mathcal{H}}^2 \leq \int_0^T \left|\psi'(1,t)\right|^2 dt. \tag{2.4}$$

Sketch of proof

Let us consider the following eigenvalue problem associated to homogeneous system

$$\begin{cases} \lambda^2\phi - \phi'' + b\lambda\psi = 0, \\ \lambda^2\psi - \psi'' - b\lambda\phi = 0, \\ \phi(0) = \phi(1) = 0, \\ \psi(0) = \psi(1) = 0 \end{cases} \tag{2.5}$$

where $b \neq 0$. For some constants C, D let

$$\phi(x) = C\sin(n\pi x), \qquad \psi(x) = D\sin(n\pi x). \tag{2.6}$$

Then, we have

$$\lambda^4 + \lambda^2(2(n\pi)^2 + b^2) + (n\pi)^4 = 0. \tag{2.7}$$

Sketch of proof

We have the following asymptotic behavior

- First branch

$$\lambda_{1,n} = in\pi + i\frac{b}{2} + i\frac{b^2}{8n\pi} + \frac{O(b^4)}{n^3},$$

Sketch of proof

We have the following asymptotic behavior

- First branch

$$\lambda_{1,n} = in\pi + i\frac{b}{2} + i\frac{b^2}{8n\pi} + \frac{O(b^4)}{n^3},$$

- second branch

$$\lambda_{2,n} = in\pi - i\frac{b}{2} + i\frac{b^2}{8n\pi} + \frac{O(b^4)}{n^3},$$

- corresponding eigenfunctions

$$\varphi_{1,n} = \frac{\sin(n\pi x)}{n\pi}, \qquad \psi_{1,n} = \frac{-i\sin(n\pi x)}{n\pi}, \tag{2.8}$$

$$\varphi_{2,n} = -\frac{i\sin(n\pi x)}{n\pi}, \quad \psi_{2,n} = \frac{\sin(n\pi x)}{n\pi}. \tag{2.9}$$

Sketch of proof

The two branches of eigenvalues of $\mathcal{A}$ satisfy an uniform gap condition

$$\gamma := \inf_{m,n} |\lambda_{1,m} - \lambda_{2,n}| > 0, \tag{2.10}$$

We set the eigenfunctions of the operator $\mathcal{A}$ as

$$\left\{ \begin{array}{l} E_{1,n} = (\varphi_{1,n}, \lambda_{1,n}\varphi_{1,n}, \psi_{1,n}, \lambda_{1,n}\psi_{1,n}), \\ E_{2,n} = (\varphi_{2,n}, \lambda_{2,n}\varphi_{2,n}, \psi_{2,n}, \lambda_{2,n}\psi_{2,n}). \end{array} \right. \tag{2.11}$$

Then we have

$$(\varphi_0, \varphi_1, \psi_0, \psi_1) = \sum_{n \neq 0} (\alpha_{1,n} E_{1,n} + \alpha_{2,n} E_{2,n}).$$

Finally

$$\int_0^T \left|\psi'(1,t)\right|^2 dt \geq c \sum_{n \neq 0} (|\alpha_{1,n}|^2 + |\alpha_{2,n}|^2).$$

This yields the inequality (2.4).

Exact controllability

We can now state the following result.

Theorem

Assume that $a = 1$, there exists no $k \in \mathbb{Z}$ and T satisfies (2.3). Let

$$(u_0, u_1, y_0, y_1) \in L^2(0,1) \times H^{-1}(0,1) \times L^2(0,1) \times H^{-1}(0,1).$$

Then there exists a control function $v \in L^2(0,T)$ such that the solution of the non homogenous system (2.1) satisfies he null final conditions:

$$u(x,T) = u_t(x,T) = y(x,T) = y_t(x,T).$$

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Thanks for your Attention!