Stability of non-conservative systems

Valenciennes, July 4th - July 7th, 2016

Qualitative propagation and decay patterns of frequency band limited signals in interacting media

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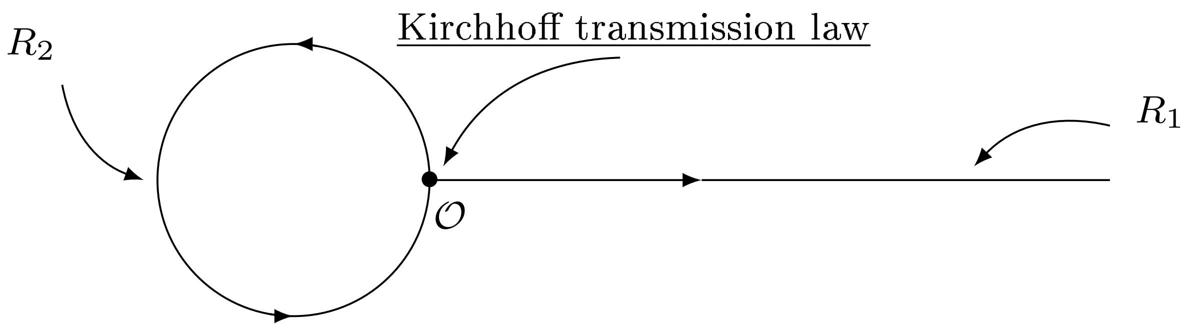
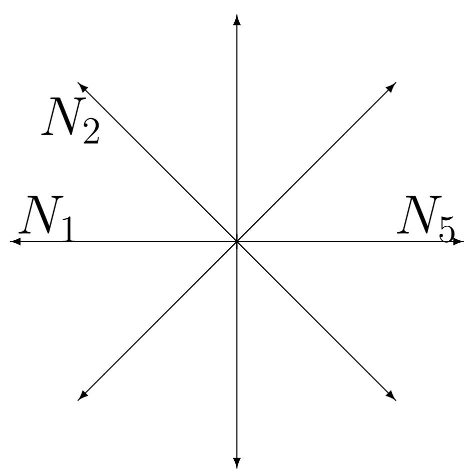
Geometries:

- Line

- Tadpole



- Star shaped network, 2 problems



Overview: Problems

	Equation	Potential	Medium	Initial condition	Publications
A	Schrödinger	——	line	frequency band, ∃ singular frequency	Dewez/FAM Math. Meth. Appl. Sci. 2016 Dewez, arXiv 2016
B	Schrödinger	——	tadpole	high frequency cutoff	Ammari/Nicaise/FAM arXiv 2015
C	Schrödinger	sufficiently localized	star-shaped network	low frequency cutoff	Ammari/Nicaise/FAM Port. Math. 2016
D	Klein-Gordon	semi-infinite different on branches	star-shaped network	frequency band	Haller-Dintelmann/Régnier/FAM Operator Theory 2012 + 2013 J. Evol. Eq. 2012

Overview: Equations

	Problem	Parame-trizations	Equations, $t \geq 0$
A	Schrödinger line	$\mathbb{R}$	$\left[i\partial_t + \partial_x^2\right] u(t,x) = 0,\ x \in \mathbb{R}$ $u(0,x) = u_0(x),\ x \in \mathbb{R}, \quad \mathcal{F}u_0$ has a singularity
B	Schrödinger tadpole	$R_1 \cong [0,\infty)$ $R_2 \cong [0,L]$	$[i\partial_t - \partial_x^2 + V_j(x)]u_j(t,x) \ = \ 0, \qquad x \in R_j, j = 1,2$ $(T_0)\ u_1(t,0) = u_2(t,0)$ $(T_1)\ \sum_{j=1}^{2} \partial_x u_j(t,0^+) - \partial_x u_2(t,L^-) \ = \ 0,$ $u_j(0,x) = u_{0,j}(x)$
C	Schrödinger star-shaped network	$N_j \cong [0,\infty)$ $j = 1,\ldots n$	$[i\partial_t - \partial_x^2 + V_j(x)]u_j(t,x) \ = \ 0, \qquad x \in N_j, j = 1,.,n$ $(T_0)\ u_1(t,0) = u_2(t,0) = \ldots = u_n(t,0),$ $(T_1)\ \sum_{j=1}^{n} c_j \partial_x u_j(t,0^+) \ = \ 0,$ $u_j(0,x) = u_{0,j}(x)$
D	Klein-Gordon star-shaped network	$N_j \cong [0,\infty)$ $j = 1,\ldots n$	$[\partial_t^2 - c_j\partial_x^2 + a_j]u_j(t,x) \ = \ 0, \qquad x \in N_j, j = 1,.,n$ $(T_0)\ u_1(t,0) = u_2(t,0) = \ldots = u_n(t,0),$ $(T_1)\ \sum_{j=1}^{n} c_j \partial_x u_j(t,0^+) \ = \ 0,$ $u_j(0,x) = u_{0,j}(x),\ \partial_t u_j(0,x) = v_{0,j}(x)$

Overview: Applications

	Problem	**Applications**
A	Schrödinger line	Quantum mechanics: free particle with a preferred value for the momentum
B	Schrödinger tadpole	Mathematical interest: simplest network with loop Quantum mechanics: particle in tadpole world, local pattern in molecules
C	Schrödinger star-shaped network	Quantum mechanics: simplified model of electrons in molecules close in to an atomic nucleus wave guides of nano tubes
D	Klein-Gordon star-shaped network	Classical wave theory: simplified models of networks of transmission lines, wave guides a_j large $\rightsquigarrow$ good conductor, bad medium for waves very good test setting for the study of the dynamics of tunnel effect

Principles:

- Semi-infinite geometry $\rightsquigarrow$ local study without reflections
- Initial conditions in frequency bands
  - $\rightsquigarrow$ propagation speed of wave packets in interval
  - $\rightsquigarrow$ detectable spatial propagation patterns

Functional analytic reformulations

	Problem	Operator
A	Schrödinger line	Space: $H = L^2(\mathbb{R})$ Operator: $A : D(A) \to H$ with $D(A) = H^2(\mathbb{R})$ and $Au = -\partial_x^2 u$ Equation: $i\dot{u} - Au = 0$
B	Schrödinger tadpole	Space: $H = L^2(R_1) \times L^2(R_2)$ Operator: $A : D(A) \to H$ with $D(A) = H^2(\mathbb{R})$ and $Au = -\partial_x^2 u$ Equation: $i\dot{u} + Au = 0$
C	Schrödinger star-shaped network	Space: $H = \prod_{k=1}^n L^2(N_k)$ Operator: $A : D(A) \to H$ with $D(A) = \{(u_k)_{k=1,\ldots,n} \in \prod_{k=1}^n H^2(N_k) : (u_k)_{k=1,\ldots,n} \text{ sat. } (T_0) \text{ and } (T_1)\}$ $A((u_k)_{k=1,\ldots,n}) = (\ [-\partial_x^2 + V_k(x)]u_k\ )_{k=1,\ldots,n}$ Equation: $i\dot{u} + Au = 0$
D	Klein-Gordon star-shaped network	Space: $H = \prod_{k=1}^n L^2(N_k)$ Operator: $A : D(A) \to H$ with $D(A) = \{(u_k)_{k=1,\ldots,n} \in \prod_{k=1}^n H^2(N_k) : (u_k)_{k=1,\ldots,n} \text{ sat. } (T_0) \text{ and } (T_1)\}$ $A((u_k)_{k=1,\ldots,n}) = (A_k u_k)_{k=1,\ldots,n} = (-c_k \cdot \partial_x^2 u_k + a_k u_k)_{k=1,\ldots,n}$ Equation: $\ddot{u} + Au = 0$

Functional analytic solution formulas

	Problem	Solution formula	Time invariant
A	Schrödinger line	$u(t) = f(t,A)u_0$ $f(t,A) = e^{-itA}$	$\Vert u(t)\Vert_H$
B	Schrödinger tadpole	$u(t) = f(t,A)u_0$ $f(t,A) = e^{itA}$	$\Vert u(t)\Vert_H$
C	Schrödinger star-shaped network	$u(t) = f(t,A)u_0$ $f(t,A) = e^{itA}$	$\Vert u(t)\Vert_H$
D	Klein-Gordon star-shaped network	$u(t) = f_1(t,A)u_0 + f_2(t,A)u_0$ $f_1(t,A) = \cos(\sqrt{A}t)$ $f_2(t,A) = (\sqrt{A})^{-1}\sin(\sqrt{A}t)v_0$	$E(u(t,\cdot)) = (Au,u)_H + \Vert\dot{u}(t)\Vert_H$

- The operator $A : D(A) \to H$ is selfadjoint
- Case A,B: $\sigma(A) = \sigma_{ac}(A) = [0,\infty)$
- Case C: $\int_{R_k} |V_k(x)|(1+|x|^2)^{1/2}\,dx < \infty \Rightarrow \sigma(A) = [0,\infty) \cup \{\text{neg. ev.}\}$
- Case D: $0 < c_j, j = 1,\dots,n$ and $0 < a_1 \leq a_2 \leq \dots \leq a_n$
  $\Rightarrow \sigma(A) = \sigma_{ac}(A) = [a_1,\infty)$

Strategy:

- Stationary problem
$$AF_\lambda = \lambda F_\lambda$$
find a 'complete' family of generalized eigenfunctions $\{F_\lambda | \lambda \in \sigma(A)\}$.
- Show that
$$(Vf)(\lambda) = \int_N \overline{F_\lambda(x)} f(x) dx$$
is a spectral repr. of H relative to A and construct a formula for V^{-1}. Tool: Stone's formula
$(h(H)E(a,b)f, g)_{\mathcal{H}} = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \frac{1}{2i\pi} \left(\int_{a+\delta}^{b-\delta} [h(\lambda) R(\lambda - i\varepsilon, H) - R(\lambda + i\varepsilon, H)] \right.$
- Functional calculus
$$f(A)u = V^{-1}[f(\lambda)(Vu)(\lambda)]$$
- Solution formula
$$u(t) = f(t, A)u_0 = V^{-1}[f(t, \lambda)(Vu)(\lambda)]$$
- Solutions in frequency bands
$$\chi_{[a,b]}(A)u(t) = V^{-1}[\chi_{[a,b]}(\lambda) f(t, \lambda)(Vu_0)(\lambda)] = f(t, A)\chi_{[a,b]}(A)u_0$$
- Solution formula leads to oscillatory integrals:
Apply *the stationary phase method* or *van der Corput type estimates*

Stationary phase method. *Theorem:* (Hörmander book 1984)
Let K be a compact interval in $\mathbb{R}$, X an open neighborhood of K. Let $U \in C_0^2(K)$, $\Psi \in C^4(X)$ and $\mathrm{Im}\Psi \geq 0$ in X. If there exists $p_0 \in X$ such that $\frac{\partial}{\partial p}\Psi(p_0) = 0,\ \frac{\partial^2}{\partial p^2}\Psi(p_0) \neq 0$, and $\mathrm{Im}\Psi(p_0) = 0,\ \frac{\partial}{\partial p}\Psi(p) \neq 0,\ p \in K \setminus \{p_0\}$, then
$\left| \int_K U(p) e^{i\omega\Psi(p)} dp - U(p_0) e^{i\omega\Psi(p_0)} \left[\frac{\omega}{2\pi i} \frac{\partial^2}{\partial p^2}\Psi(p_0) \right]^{-1/2} \right| \leq C(K) \, \|U\|_{C^2(K)} \, \omega^{-1}$
for all $\omega > 0$. Moreover $C(K)$ is bounded when Ψ stays in a bounded set in $C^4(X)$.

Van der Corput type estimate. *Theorem:* (Dewez arXiv 2015)
Suppose $\psi \in \mathcal{C}^1\big(I\big) \cap \mathcal{C}^2\big(I \backslash \{p_0\}\big)$ and the existence of $\tilde{\psi} : I \longrightarrow \mathbb{R}$ such that

$$\forall\, p \in I \qquad \psi'(p) = |p - p_0|^{\rho-1}\, \tilde{\psi}(p) \ ,$$

where $\big|\tilde{\psi}\big| : I \longrightarrow \mathbb{R}$ is assumed continuous and does not vanish on I. and $\forall\, p \in (p_1, p_2] \qquad U(p) = (p - p_1)^{\mu-1}\, \tilde{u}(p)$, Moreover suppose that ψ' is monotone on $\big\{p \in I \,\big|\, p < p_0\big\}$ and $\big\{p \in I \,\big|\, p > p_0\big\}$. Then we have

$$\left| \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp \right| \leq C(U, \psi)\, \omega^{-\frac{\mu}{\rho}} \ ,$$

for all $\omega > 0$, and the constant $C(U, \psi) > 0$ is given in the proof.

A.
The free Schrödinger equation on the line, initial condition with singular frequency

Solution formula free equation on the line

$$u(t,x) = \frac{1}{2\pi} \int_{p_1}^{p_2} \mathcal{F}u_0(p)\, e^{-itp^2+ixp}\, dp \ ,$$

With

$$U(p) := \mathcal{F}u_0(p) \qquad , \qquad \psi(p) := -p^2 + \frac{x}{t}p \ .$$

the solution

$$u(t,x) = \frac{1}{2\pi} \int_{p_1}^{p_2} U(p)\, e^{it\psi(p)}\, dp \ .$$

takes the form of an oscillatory integral with respect to the large parameter t, where U is called amplitude and ψ the phase.

Idea: Use a stationary phase method to obtain an asymptotic expansion of

$$t \longmapsto \int_{p_1}^{p_2} U(p)\, e^{it\psi(p)}\, dp \ .$$

Stationary points of the phase: $\frac{d}{dp}\psi(p) = -2p + \frac{x}{t} = 0 \Leftrightarrow p = \frac{x}{2t}$.
$\rightsquigarrow$ frequency band $[p_1, p_2]$ corresponds to a cone $p_1 \leq \frac{x}{2t} \leq p_2$ in space time.

Asymptotic expansions in cones.

Theorem:

Fix $\mu \in (0,1)$ and let $p_1 < p_2$ be two finite real numbers. Suppose that u_0 satisfies $\operatorname{supp} \mathcal{F}u_0 = [p_1, p_2]$, $\mathcal{F}u_0(p_2) = 0$ and $\mathcal{F}u_0 = (p-p_1)^{\mu} f(p)$ with $f \in C^1([p_1, p_2], \mathbb{C}$. Fix $\delta \in \big(\max\{\mu, \frac{1}{2}\}, 1\big)$ and $\varepsilon > 0$ such that $p_1 + \varepsilon < p_2$.

We define the cone $\mathfrak{C}_{\varepsilon}(p_1, p_2)$ as follows

$$p_1 + \varepsilon \leq \frac{x}{2t} \leq p_2 \ .$$

Then for all $(t,x) \in \mathfrak{C}_{\varepsilon}(p_1, p_2)$, there exist complex numbers $K_{\mu}(t, x, u_0)$, $H(t, x, u_0) \in \mathbb{C}$ and a constant $c(u_0, \varepsilon, \delta) \geq 0$ satisfying

$$\boxed{\left|u(t,x) - H(t,x,u_0)\, t^{-\frac{1}{2}} - K_{\mu}(t,x,u_0)\, t^{-\mu}\right| \leq \textstyle\sum_{k=1}^{8} R_k(u_0, \varepsilon)\, t^{-\alpha_k} \ ,}$$

where $R_k(u_0, \varepsilon) \geq 0$ $(k = 1, ...8)$ are constants independent from t and x, and $\alpha_k > \max\left\{\mu, \frac{1}{2}\right\}$.

Explicit Formulas.

– $H(t,x,u_0) = \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, f\left(\frac{x}{2t}\right)\left(\frac{x}{2t}-p_1\right)^{\mu-1}$

– $K_\mu(t,x,u_0) = \frac{\Gamma(\mu)}{2^{\mu+1}\pi}\, e^{i\frac{\pi\mu}{2}}\, e^{i(-tp_1^2+xp_1)}\, f(p_1)\left(\frac{x}{2t}-p_1\right)^{-\mu}$

– $R_k(u_0,\varepsilon) = r_k(u_0)(\frac{x}{2t}-p_1)^{-\beta_k}$, formulas for $r_k(u_0), \alpha_k, \beta_k$ available.

Method.

– The proof uses a improved and refined version of the stationary phase formula sketched by A. Erdélyi.

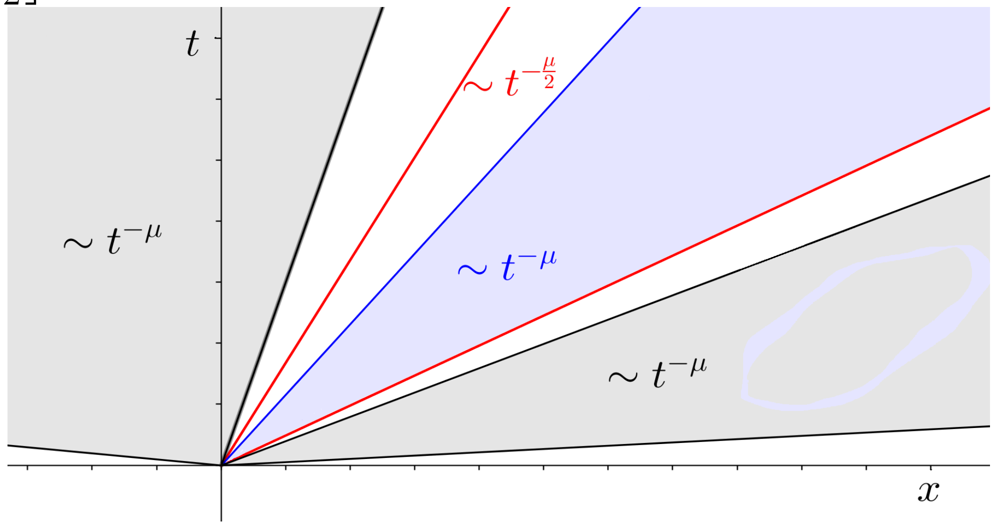
– This formula is based on complex analysis in one variable. The integration path is deformed in regions with less oscillations.

– Smooth cut-off functions (used in most versions) are replaced by characteristic functions.

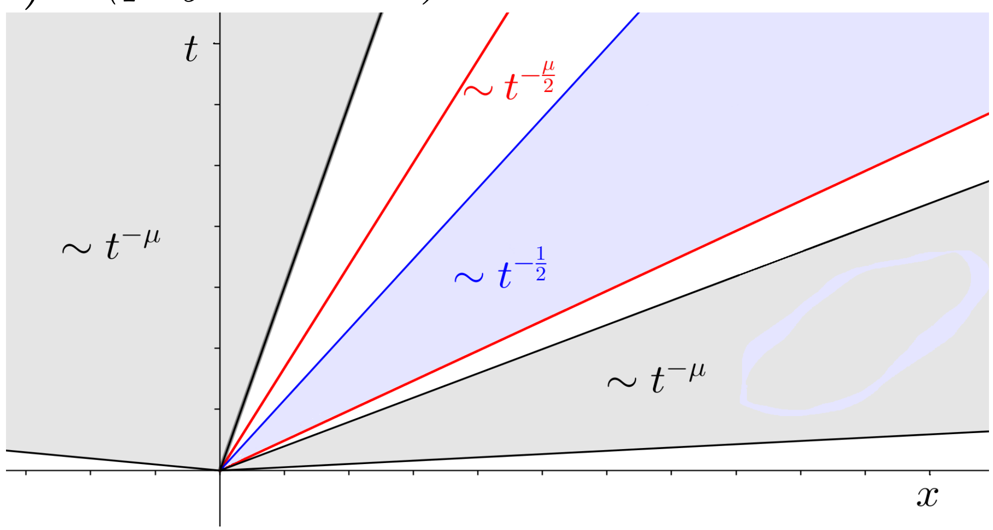
– Advantages: stationary points of real order, singular amplitudes, lossless error estimates.

Summary.

– Case $\mu \in \left(0, \frac{1}{2}\right]$:



– Case $\mu \in \left(\frac{1}{2}, 1\right)$: *(physical case)*



L^2-norm estimate inside $\mathfrak{C}_{\varepsilon_1,\varepsilon_2}(p_1,p_2)$

The preceeding Theorem leads to an estimate of the L^2-norm inside the cone when $\mu \in \left(\frac{1}{2}, 1\right)$.

Theorem:
Suppose that u_0 satisfies Condition $(C_{p_1,p_2,\mu})$ with $\mu \in \left(\frac{1}{2}, 1\right)$. Fix $\varepsilon_1, \varepsilon_2 > 0$ such that $p_1 + \varepsilon_1 < p_2 - \varepsilon_2$. Then there exists a constant $c(u_0, \varepsilon_1, \varepsilon_2) \geq 0$ such that for all $t \geq 1$,

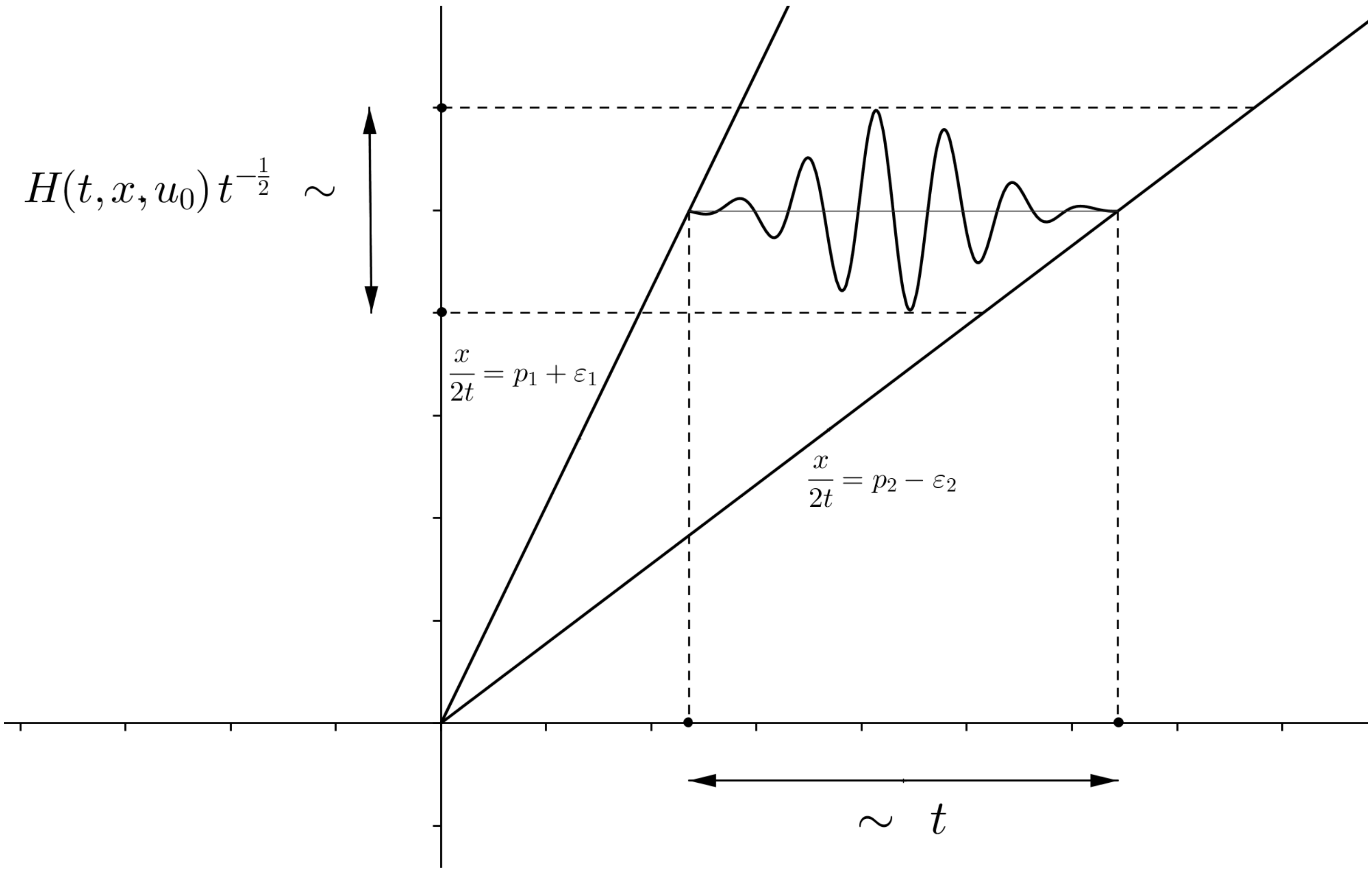
$$\left| \|u(t,.)\|_{L^2(I_t)} - \frac{1}{\sqrt{2\pi}} \, \|\mathcal{F}u_0\|_{L^2(p_1+\varepsilon_1, p_2-\varepsilon_2)} \right| \leq c(u_0, \varepsilon_1, \varepsilon_2)\, t^{\frac{1}{2}-\mu} \ ,$$

where

$$I_t := \left[\, 2\,(p_1 + \varepsilon_1)\, t,\ 2\,(p_2 - \varepsilon_2)\, t \right] \ .$$

– According to Plancherel's Theorem, a large part of the norm is concentrated in the cone
– The probability amplitude behaves time-asymptotically as a laminar flow.

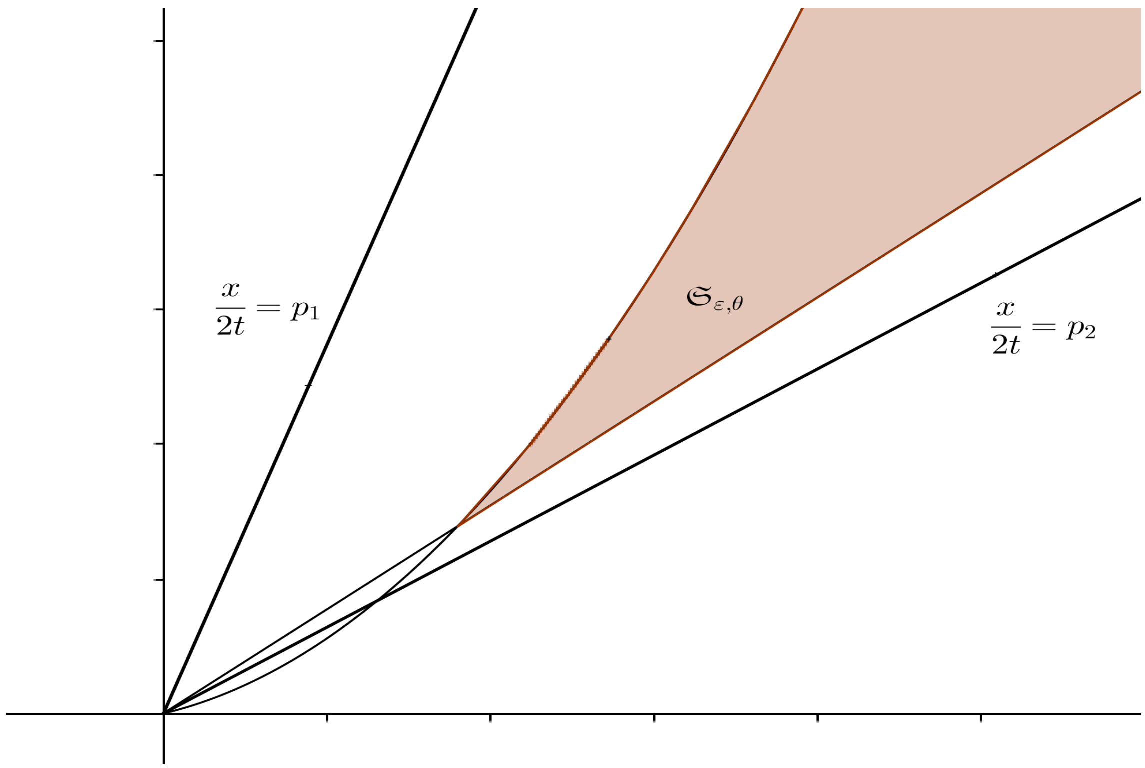
Idea of the proof



Estimates in regions bounded by curves

Let the region $\mathfrak{R}_\varepsilon$ be described as follows:

$$(t,x) \in \mathfrak{R}_\varepsilon \qquad \Longleftrightarrow \qquad \begin{cases} \dfrac{x}{2t} - p_1 \geq t^{-\varepsilon} \ , \\ x < 2p_2\, t \ , \\ t > \big(2(p_2 - p_1)\big)^{-\frac{1}{\varepsilon}} \ . \end{cases}$$



Estimates in regions defined by curves

Suppose that u_0 satisfies Condition $(C_{p_1,p_2,\mu})$.
Fix $\delta \in \left[\frac{\mu+1}{2}, 1\right)$ and $\varepsilon \in \left(0, \delta - \frac{1}{2}\right)$.
Then there exist three constants $C_1(u_0), C_2(u_0), C_3(u_0) > 0$ such that for all $(t,x) \in \mathfrak{R}_\varepsilon$, the following estimates hold:

$$\big|u(t,x)\big| \leq \begin{cases} C_1(u_0)\, t^{-\frac{1}{2}+\varepsilon(1-\mu)} & , \quad \text{if } \mu > \frac{1}{2} \,, \\ C_2(u_0)\, t^{-\frac{1}{2}+\frac{\varepsilon}{2}} & , \quad \text{if } \mu = \frac{1}{2} \,, \\ C_3(u_0)\, t^{-\mu+\varepsilon\mu} & , \quad \text{if } \mu < \frac{1}{2} \,. \end{cases}$$

The decay rates are attained on the left boundary of $\mathfrak{R}_\varepsilon$.

<u>Remarks:</u> When ε tends to the critical value $\frac{1}{2}$,

- the decay rates tend to $t^{-\frac{\mu}{2}}$,
- the constants tend to infinity.

Global estimate (F. Dewez, 2015)

$$\|u(t,.)\|_{L^\infty(\mathbb{R})} \leq \text{ const } t^{-\frac{\mu}{2}} \; .$$

– Cannot be derived from an expansion to one term: when the critical direction in space-time is attained, the expansion changes its nature. The coefficient blows up in the vicinity of the critical direction.
– Optimal
– Proof uses a van der Corput type estimate for oscillating integrals with an amplitude with an integrable singularity (F. Dewez, 2015).
– physical case: $\mu \in (\frac{1}{2}, 1) \to u_0 \in L^2(\mathbb{R})$. By Strichartz:

$$\|u(t,.)\|_{L^\infty(\mathbb{R})} \leq const\; t^{-\frac{1}{4}} \; .$$

The above result is more precise: $\frac{\mu}{2} \in (\frac{1}{4}, \frac{1}{2})$.
– $\mu \in (0, \frac{1}{2}] \to u_0 \notin L^2(\mathbb{R})$.
In any case $u_0 \notin L^1(\mathbb{R}) \rightsquigarrow$ known results not applicable.

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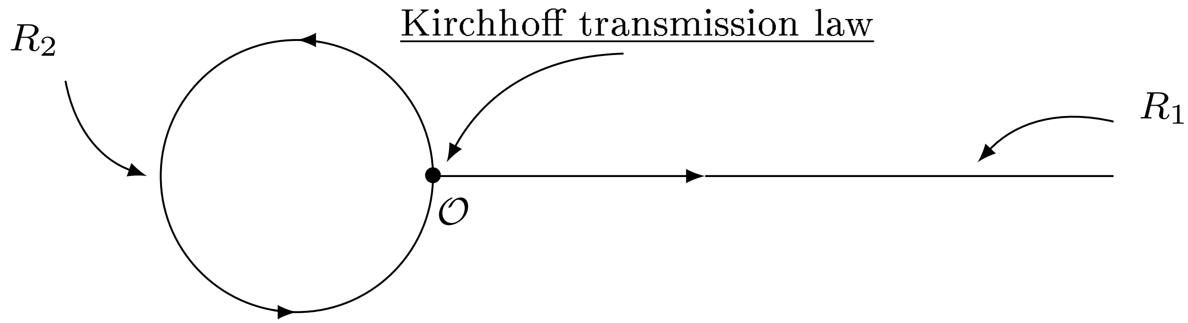
B.
The free Schrödinger equation on tadpole network

- Tadpole



Results

Theorem:

Dispersive estimate: For all $t \neq 0$,

$$\left\| e^{itA} P_{ac} \right\|_{L^1(\mathcal{R}) \to L^\infty(\mathcal{R})} \le C \ |t|^{-1/2} ,$$

where C is a positive constant independent of L and t, $P_{ac}f$ is the projection onto the absolutely continuous spectral subspace and
$L^1(\mathbb{R}) = \prod_{k=1}^2 L^1(R_k),\ L^\infty(\mathbb{R}) = \prod_{k=1}^2 L^\infty(R_k)$.
Scale invariance:
If the above inequality holds for a certain constant C and circumference L_0 then it holds for all circumferences with the same C.

Theorem:

Dispersive perturbation estimate: Let A_0 be the negative laplacian on the half line with Neumann boundary conditions. Let $0 \le a < b < \infty$. Let $u_0 \in \mathcal{H} \cap L^1(R_1)$ such that

$$\text{supp } u_0 \subset R_1 \ . \tag{1}$$

Then for all $t \neq 0$, we have

$$\begin{aligned} &\| \ e^{itA} \chi_{(a,b)}(A) u_0 \ - \ e^{itA_0} \chi_{(a,b)}(A_0) u_0 \ \|_{L^\infty(R_1)} \\ &\qquad \le \ t^{-1/2} L \ 2\sqrt{2} \ \Big(4(2\sqrt{b} - \sqrt{a}) + L(b-a) \Big) \ \|u_0\|_{L^1(R_1)} \, . \end{aligned}$$

Spectral Theory.

Theorem:

Take $f, g \in \mathcal{H}$ with a compact support and let $0 < a < b < +\infty$. Then for any holomorphic function h on the complex plane, we have

$$(h(A)\chi_{(a,b)}(A)f, g)_{\mathcal{H}} = -\frac{1}{\pi}\int\limits_{\mathcal{R}} \Big(\int_{(a,b)} h(\lambda) \int\limits_{\mathcal{R}} f(x')\mathrm{Im}K_c(x, x', \lambda)\; dx'\; d\lambda \Big) \bar{g}(x) dx$$

$$+ \sum_{k\in\mathbb{N}*:a<\lambda_{2k}^2<b} h(\lambda_{2k}^2)(f, \varphi^{(2k)})_{\mathcal{H}}(\varphi^{(2k)}, g)_{\mathcal{H}},$$

where an explicit expression for $K_c(x, x', \lambda)$ available and for all $k \in \mathbb{N}^*$, the number $\lambda_{2k}^2 = \frac{4k^2\pi^2}{L^2}$ is an eigenvalue of H of the associated eigenvector $\varphi^{(2k)} \in D(A)$ given by

$$\varphi_1^{(2k)} = 0 \text{ in } R_1, \varphi_2^{(2k)}(x) = \frac{\sqrt{2}}{\sqrt{L}} \sin(\lambda_{2k}x), \forall x \in R_2.$$

Corollary:

$$\sigma_{ac}(A) = [0, \infty), \quad \sigma_{pp}(A) = \{\lambda_{2k}^2, k \in \mathbb{N}^*\}, \quad \sigma_{sc}(A) = \emptyset.$$

Interpretation:

– $\sigma_{ac}(A) \rightsquigarrow$ interaction circle - half line.
– $\sigma_{pp}(A) \rightsquigarrow$ states confined in circle.
– The terms of the series appear by the residue theorem.

Perturbation.

Theorem:

Let $\left(e^{itA}\chi_{(a,b)}(A)P_{ac}\right)(x,y)$ and $\left(e^{itA_0}\chi_{(a,b)}(A_0)\right)(x,y)$ be the kernels of the operator groups in the brackets. For $0 \le a < b < \infty$ and $x, y \in R_1 \cong (0,\infty)$ we have

i)

$$\left(e^{itA}\chi_{(a,b)}(A)P_{ac}\right)(x,y) \; - \; \left(e^{itA_0}\chi_{(a,b)}(A)(A_0)\right)(x,y)$$
$$= \int_{\sqrt{a}}^{\sqrt{b}} e^{i(t\mu^2+\mu(x+y))}\frac{4(1-e^{i\mu L})}{e^{i\mu L}-3}e^{i\mu(x+y)}d\mu$$

ii)

$$\left|\left(e^{itA}\chi_{(a,b)}(A)P_{ac}\right)(x,y) \; - \; \left(e^{itA_0}\chi_{(a,b)}(A)(A_0)\right)(x,y)\right|$$
$$\le t^{-1/2}L\, 2\sqrt{2}\left(4(2\sqrt{b}-\sqrt{a})+L(b-a)\right)$$

Corollary:

Let $0 \le a < b < \infty$. Let $u_0 \in H \cap L^1(R_1)$ such that

$$\text{supp } u_0 \subset R_1 \; .$$

Then we have

$$\| \; e^{itA}\chi_{(a,b)}(A)u_0 \; - \; e^{itA_0}\chi_{(a,b)}(A_0)u_0 \; \|_{L^\infty(R_1)}$$
$$\le t^{-1/2}L\, 2\sqrt{2}\left(4(2\sqrt{b}-\sqrt{a})+L(b-a)\right) \; \|u_0\|_{L^1(R_1)}$$

Interpretation:

– Rescaled by decay:
Solution on queue with upper frequency cutoff
$\rightarrow$ solution of the half-line Neumann problem with the same upper frequency cutoff
(support of initial condition in queue).

– The upper frequency cutoff introduces in physical terms an upper limit for the (group) velocity of wave packets and thus a lower limit for the localization of wave packets.

– This destroys the scale invariance: low frequency signals do not see the head

– Technically this estimate can be reduced to the inequality

$$| 1 - e^{i\mu L} | \leq \mu L$$

$\rightsquigarrow$ couples circumference and frequency.

– Interpretation of formula i) in the Theorem:

$$\frac{1}{e^{i\mu L} - 3} = -\frac{1}{3} \sum_{k=0}^{+\infty} \frac{e^{ik\mu L}}{3^k},$$

is a series representation of the difference of the solutions of the tadpole problem on its queue and the half-line Neumann problem:
signals passing from the head of the tadpole into its queue after k cycles around the head.

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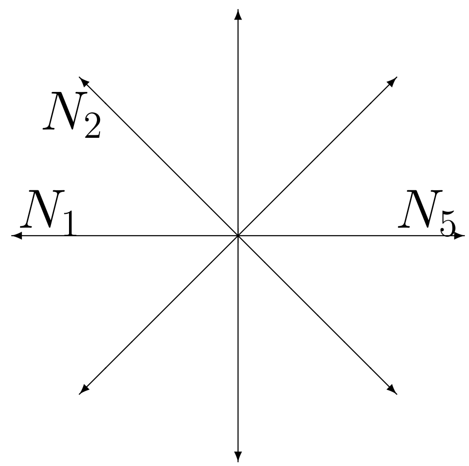
C.

The Schrödinger equation with localized potential on a star shaped network

- Star shaped network



Dispersive Estimate:

Consider the space $L^1_s(\mathcal{R}) := \{\phi = (\phi_1, \ldots, \phi_N) : \mathcal{R} \to \mathbb{C} \mid \|\phi\|_{L^1_s(\mathcal{R})} := \sum_{k=1}^N \int_{R_k} |\phi_k(x)|(1+|x|^2)^{s/2} < \infty \, dx\}$.

Theorem: (AM/Ammari/Nicaise)
Let $V = (V_k)_{k=1,\ldots,n} \in L^1_\gamma(\mathcal{R})$ be real valued, with $\gamma > 5/2$ and assume a (generic) condition of non resonance. Then for all $t \neq 0$,

$$\boxed{\left\|e^{itA} P_{ac}(A)\right\|_{L^1(\mathcal{R}) \to L^\infty(\mathcal{R})} \leq C \ |t|^{-1/2}}$$

where C is a positive constant and $P_{ac}(A)$ is the projection onto the absolutely continuous spectral subspace.

– Free particle in $\mathbb{R}^n$: Reed/Simon II ($t^{-n/2}$)
– Expresses dynamics of the uncertainty relation.
– Particle submitted to potential on the line: R. Weder 2000, M. Goldberg, W. Schlag (2004), on the half-line: R. Weder 2003.
– Spectral theory: reduction to sacattering theory on the line: Goldberg/Schlag, on the star shaped network: R. Haller-Dintelmann/V. Régnier/FAM 2008
– Dispersion: treat differently low and high frequencies.

High energy perturbation estimate:

Theorem: (AM/Ammari/Nicaise)

Under the assumptions of the preceding theorem and for $\lambda_0 > \lambda_*$ we have

$$\|e^{itA}\chi_{\lambda_0}(A)\|_{1,\infty} \leq (a + b\frac{\|V\|_1}{\sqrt{\lambda_0}})|t|^{-1/2}, t \neq 0,$$

$$\boxed{\|e^{itA}\chi_{\lambda_0}(A) - e^{itA_0}\chi_{\lambda_0}(A_0)\|_{1,\infty} \leq b\frac{\|V\|_1}{\sqrt{\lambda_0}}|t|^{-1/2}, t \neq 0\,.}$$

Here χ_{λ_0} is smoothly cutting off the frequencies below λ_0. Expressions for a, b in terms of the cutoff function but independent of λ_0 as well for λ_* are given in the article.

In particular we have for any $f \in L^1(\mathcal{R})$ that

$$e^{itA}\chi_{\lambda_0}(A)f \to e^{itA_0}\chi_{\lambda_0}(A_0)f \text{ for } \lambda_0 \to \infty$$

uniformly on $\mathcal{R}$ for every fixed $t > 0$.

Observation: Rescaled by the decay, the **interacting solution is close to the free solution** in the high frequencies, if

– the potential is small, or

– the cutoff frequency is high.

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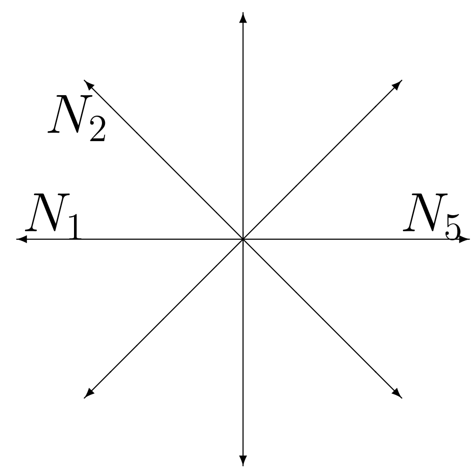
D.
The Klein-Gordon equation with potential steps on a star shaped network

- Star shaped network



Generalized Eigenfunctions:

For $\lambda \in \mathbb{C}$, $j \in \{1, \ldots, n\}$, let $F_\lambda^{\pm,j} : N \to \mathbb{C}$ be defined by

$$F_\lambda^{\pm,j}(x) = \begin{cases} \cos(\xi_j(\lambda)x) \pm i s_j(\lambda) \sin(\xi_j(\lambda)x), & \\ \exp(\pm i \xi_k(\lambda)x), & \text{for } k \neq j. \end{cases}$$

for $x \in \overline{N_k}$. Here

$$\xi_k(\lambda) := \sqrt{\frac{\lambda - a_k}{c_k}} \qquad \text{and} \qquad s_k(\lambda) := -\frac{\sum_{l \neq k} c_l \xi_l(\lambda)}{c_k \xi_k(\lambda)}.$$

– $F_\lambda^{\pm,j}$ satisfies $(T_0), T(_1)$ and $Au = \lambda u$
– $F_\lambda^{\pm,j} \notin H \rightsquigarrow$ "generalized" eigenfunctions.

$\rightsquigarrow$ vectorvalued transform

$$(Vg)(\lambda) := \int_N \overline{F_\lambda(x')} g(x') dx'$$

where

$$F_\lambda(x') := \left(F_\lambda^{-,1}(x'), \ldots, F_\lambda^{-,n}(x') \right)^T$$

<u>Problem:</u>
Find space setting such that V is an isometry and a formula for V^{-1} :

$$(V^{-1}G)(x) = \int_{\sigma(A)} F_\lambda(x)^T q(\lambda) G(\lambda) d\lambda$$

Spectral Representation:

Definition:
For $\lambda \in \mathbb{R}$ and $l \in \{1, \dots, n\}$ such that $a_l < \lambda < a_{l+1}$ where $a_{n+1} = \infty$ we define

$$q(\lambda) = \frac{1}{|w(\lambda)|^2} \begin{pmatrix} c_1\xi_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & c_l\xi_l & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}$$

and $q(\lambda) = 0$ for $\lambda \le a_1$.

Theorem:
For $h \in C(\mathbb{R})$ we have
i)

$$\begin{aligned} h(A)E(a,b)g &= \int_a^b h(\lambda) F_\lambda^T q(\lambda)(Vg)d\lambda \\ &= V^{-1}[h\mathbf{1}_{[a,b]}(Vg)],\ g \in H \end{aligned}$$

ii) $V : H \to L^2_q$ isometry, spectral representation
iii) $u \in D(A^j) \Leftrightarrow \lambda \mapsto \lambda^j (Vu)_k(\lambda) \in L^2((a_k, +\infty), q_k)$, for all $k = 1, \dots, n$.

Multiple tunnel effect:

Denoting $P_j = \left(\begin{array}{c|c} I_j & 0 \\ \hline 0 & 0 \end{array}\right)$, where I_j is the $j \times j$ identity matrix, for $\lambda \in (a_j, a_{j+1})$ it holds: $F_\lambda^T q(\lambda) F_\lambda = (P_j F_\lambda)^T q(\lambda)(P_j F_\lambda)$ and

$$P_j F_\lambda = \begin{pmatrix} \left(+, *, *, \ldots, *, e^{-|\xi_{j+1}|x}, \ldots, e^{-|\xi_n|x}\right) \\ \left(*, +, *, \ldots, *, e^{-|\xi_{j+1}|x}, \ldots, e^{-|\xi_n|x}\right) \\ \left(*, *, +, \ldots, *, e^{-|\xi_{j+1}|x}, \ldots, e^{-|\xi_n|x}\right) \\ \vdots \\ \left(*, *, \ldots, *, +, e^{-|\xi_{j+1}|x}, \ldots, e^{-|\xi_n|x}\right) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2}$$

Here $*$ means $e^{-i\xi_k(\lambda)}$ and $+$ means $\cos(\xi_k(\lambda)x) - is_k(\lambda)\sin(\xi_k(\lambda)x)$ in the k-th column for $k = 1, \ldots, j$.

$\rightsquigarrow$ *Tunnel effect in the last $(n-j)$ branches with different exponential decay rates*

L^∞–time asymptotics:

– Solution formula

$$u = \cos(\sqrt{A}t)u_0 + (\sqrt{A})^{-1}\sin(\sqrt{A}t)v_0 =$$
$$= V^{-1}[\cos(\sqrt{\lambda}t)(Vu_0)(\lambda)] + V^{-1}[(\sqrt{\lambda})^{-1}\sin(\sqrt{\lambda}t)(Vv_0)(\lambda)]$$

is now concrete!

– Model case: $n = 2$, $c_1 = c_2 = 1$, $v_0 \equiv 0$.

Theorem: Let u_+ be the solution in N_2 moving away from 0. Suppose $0 < \alpha < \beta < 1$ and $\psi \in C_c^2((\alpha, \beta))$ with $\|\psi\|_\infty = 1$. Choose $u_0 \in H$ with $(Vu_0)_2 \equiv 0$ and $(Vu_0(\lambda))_1 = \psi(\lambda - a_2)$ Then there is a constant $C(\psi, \alpha, \beta)$ independent of a_1 and a_2, such that for all $t \in \mathbb{R}^+$ and all $x \in N_2$ with

$$\sqrt{\frac{a_2+\beta}{\beta}} \le \frac{t}{x} \le \sqrt{\frac{a_2+\alpha}{\alpha}}$$

we have

$$\left|u_+(t,x) - H(t,x,u_0)\cdot t^{-1/2}\right| \le C(\psi,\alpha,\beta)\cdot t^{-1}$$

with

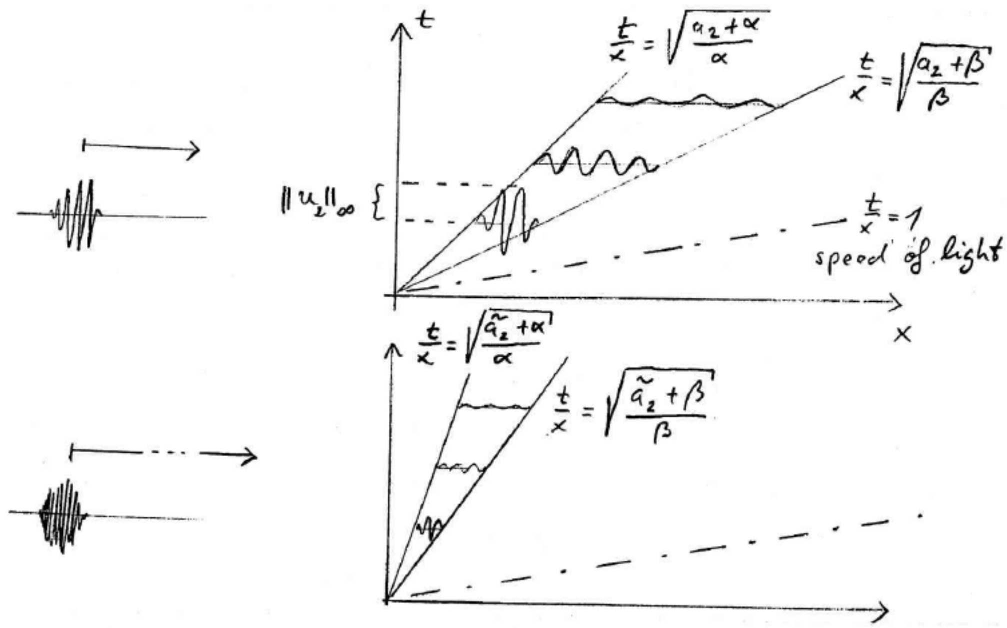
$$\left|H(t,x,u_0)\right| \le \sqrt{2\pi}\frac{\sqrt{\beta}(a_2+\beta)^{3/4}}{\sqrt{a_2}\sqrt{a_2-a_1+\beta}} \sim \sqrt{2\pi\beta}a_2^{-1/4},\ a_2 \to \infty.$$

Estimate from below $\sim \sqrt{2\pi\alpha}a_2^{-1/4}$, $a_2 \to \infty$, If $\psi \ge m > 0$ on sub-interval

Interpretation as electromagnetic wave propagation

a_2 small

$\tilde{a}_2$ big



– Frequency band at constant distance above cutoff frequency
$\rightsquigarrow$ Wavelength constant in N_2

– Growing cutoff frequency $\sqrt{a_2}$
$\rightsquigarrow$ medium in N_2 is a better conductor (more metal-like = reflecting) or diameter of the wave guide N_2 decreases.

– Cone more inclined towards t-axis $\rightsquigarrow$ diminished group velocity in N_2

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Overview: Propagation features

	Problem	**Propagation features**
A	Schrödinger line	Propagation of wave packets, spatial dissociation of frequencies
B	Schrödinger tadpole	(Circle $\to$ point) + high frequency cutoff $\Rightarrow$ solution $\to$ solution on half line (Neumann conditions) with known propagation features
C	Schrödinger star-shaped network	Lower cutoff frequency $\to \infty$ $\Rightarrow$ solution $\to$ solution of free problem (with known propagation features)
D	Klein-Gordon star-shaped network	Propagation features as in A. + Exact impact of coefficients and frequency band on reflection, splitting, and propagation

Challenges:

- Generalized notion of reflexion and transmission for general (localized) potetials
- Nonlinear equations
- Higher space dimensions
- Higher propagation features (Ehrenfest principle)
- Functional analysis