Null controllability of hypoelliptic quadratic PDEs

Karine Beauchard

ENS Rennes, IRMAR

Valenciennes, 7 Juillet 2016
Stability of nonconservative systems

Goal

We aim at studying the **null controllability** of parabolic equations posed **on the whole space** $\mathbb{R}^n$, by means of a source term u locally distributed on an open subset $\omega \subset \mathbb{R}^n$

$$\begin{cases} (\partial_t + P)f(t,x) = u(t,x)\mathbb{1}_\omega(x)\,, & x \in \mathbb{R}^n, \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^n), \end{cases}$$

where $P = q^w(x, D_x)$ is an accretive quadratic operator. They are non self-adjoint differential operators, with explicit expression

$$x^\alpha \xi^\beta \longrightarrow \frac{1}{2}\left(x^\alpha D_x^\beta + D_x^\beta x^\alpha\right)$$

$$\forall f_0 \in L^2(\mathbb{R}^n)\,, \exists u \in L^2((0,T)\times\mathbb{R}^n) \text{ such that } f(T) = 0\,.$$

We want to understand to which extend the null controllability results known for the heat equation still hold for degenerate parabolic equations of hypoelliptic type.

Motivation : nonlinear models such as Boltzmann, Prandtl,...

For the heat equation on the whole space

$$(\partial_t - \Delta_x) f(t,x) = u(t,x) \mathbb{1}_\omega(x), \quad x \in \mathbb{R}^n,$$

no NSC on ω is known for null-controllability to hold in any positive time. We know [Miller 2005]

- **a necessary condition** : $\sup_{x\in\mathbb{R}^n} d(x,\omega) < +\infty$,
- **a sufficient condition** :

$$\exists \delta, r > 0, \forall y \in \mathbb{R}^n, \exists y' \in \omega, \quad B(y', r) \subset \omega \text{ and } |y - y'| < \delta$$

Goal : Identify classes of hypoelliptic operators for which the same null controllability result holds.

Necessary condition for observability of the heat equation

Assume that $\sup_{x\in\mathbb{R}} d(x,\omega) = +\infty$. We consider $(x_k)_{k\geq 0} \subset \mathbb{R}^n$ s.t. $a_k := \text{dist}(x_k,\omega) \underset{k\to\infty}{\longrightarrow} +\infty$ and $g(t,x) = G_{\epsilon+t}(x-x_k)$ where G_t denotes the **heat kernel**

$$G_t(y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}}, \qquad t>0,\ y\in\mathbb{R}.$$

If the **heat equation** was **observable** on ω in time $T>0$, then

$$\exists C_T>0, \forall g_0\in L^2(\mathbb{R}), \quad \int_{\mathbb{R}} |g(T,x)|^2 dx \leqslant C_T \int_0^T \int_\omega |g(t,x)|^2 dxdt$$

$$\text{i.e.} \quad \frac{c_0}{\sqrt{T+\epsilon}} = \int_{\mathbb{R}} |G_{\epsilon+T}(y)|^2 dy \leq C_T \int_\epsilon^{\epsilon+T} \int_{|y|\geq a_k} |G_t(y)|^2 dydt$$
$$\leq C_T' \int_\epsilon^{\epsilon+T} \frac{1}{\sqrt{t}} e^{-\frac{a_k^2}{2t}} dt \quad \underset{k\to\infty}{\longrightarrow} 0$$

Rk : $\frac{1}{\sqrt{\pi}} \int_{|x|\geq a} e^{-x^2} dx \leq e^{-a^2}, \quad \forall a \geqslant 0$

A previous result

Le Rousseau and Moyano [2015] proved that the Kolmogorov equation

$$\partial_t f + v.\nabla_x f - \Delta_v f = u(t,x,v)1_\omega(x,v)\,, \quad (t,x,v) \in (0,T) \times \mathbb{R}^n \times \mathbb{R}^n$$

is null controllable from $\omega = \omega_x \times \omega_v$ when both ω_x and ω_v satisfy

$$\exists \delta, r > 0, \forall y \in \mathbb{R}^n\,, \exists y' \in \omega, \quad B(y',r) \subset \omega_x \text{ and } |y-y'| < \delta$$

Strategy :

1. **Partial Fourier transform in variable** x :

$$\partial_t \hat{f} + i\xi v \hat{f} - \Delta_v \hat{f}(t,\xi,v) = 1_{\omega_v} \hat{u}(t,\xi,v)\,, \quad (\xi, v) \in \mathbb{R}^n \times \mathbb{R}^n$$

2. **Global Carleman parabolic estimates** on $(t,v) \mapsto \hat{f}(t,\xi,v)$ $\longrightarrow$ localization in variable v
3. **Lebeau-Robbiano's method** $\longrightarrow$ localization in variable x

Lebeau-Robbiano's method : presentation on the heat eq

$$(\partial_t - \Delta) f(t,x) = u(t,x) 1_\omega(x) \text{ in } \Omega \qquad f(t,.) = 0 \text{ on } \partial\Omega$$

- **Spectral inequality :** $-\Delta \phi_j = \mu_j \phi_j$ in Ω $\quad \phi_j = 0$ on $\partial\Omega$

$$\Big\| \sum_{\mu_j \leqslant \mu} \alpha_j \phi_j \Big\|_{L^2(\Omega)} \leqslant K e^{K\sqrt{\mu}} \Big\| \sum_{\mu_j \leqslant \mu} \alpha_j \phi_j \Big\|_{L^2(\omega)} \qquad \forall (\alpha_j)$$

- **Iterative procedure :** $0 = T_0 < T_1 < \ldots < T_j \rightarrow T$
  - on $[T_j, T_{j+1/2}]$, one applies a control that steers to zero $\Pi_j f(T_{j+1/2})$, the projection onto the energy levels $\leqslant \mu = 2^{2j}$

$$\|u\| \leqslant K e^{K 2^j} \|f(T_j)\| \qquad \|f(T_{j+1/2})\| \leqslant K e^{K 2^j} \|f(T_j)\|$$

  - on $[T_{j+1/2}, T_{j+1}]$, no control $\rightarrow$ dissipation

$$\|f(T_{j+1})\| \leqslant \|f(T_{j+1/2})\| e^{-2^{2j}\tau_j} \leqslant K e^{K 2^j - 2^{2j}\tau_j} \|f(T_j)\|$$

Key point : dissipation 2^{2j} $\quad >>> \quad$ spectral inequality's cst 2^j

RK : The projections Π_j need to commute with the semi-group

Lebeau-Robbiano's method for the Kolmogorov equation

$$\partial_t f + v.\nabla_x f - \Delta_v f = u(t,x,v) 1_\omega(x,v)\,, \quad (x,v) \in \mathbb{R}^n \times \mathbb{R}^n$$

Packets for the form $f(t,x,v) = \frac{1}{(2\pi)^n} \int_{-N}^{N} \hat{f}(t,\xi,v) e^{ix.\xi} d\xi$

1. **For a fixed ξ, the cost of the null controllability from** ω_v of $(t,v) \mapsto \hat{f}(t,\xi,v)$ is $\leqslant e^{c\left(1+\frac{1}{T}+\sqrt{|\xi|}\right)}$
2. **Spectral inequality :** $\exists c_1 > 0$ such that, $\forall N \geq 0$ and $g \in L^2(\mathbb{R}^n)$ with $\text{supp}(\hat{g}) \subset B(0,N)$ *(elliptic Carleman estimate)*
$$\|g\|_{L^2(\mathbb{R}^n)} \leqslant c_1 e^{c_1 N} \|g\|_{L^2(\omega_x)}$$
3. **Dissipation speed** without control *(explicit Fourier transform)*
$$\|\hat{f}(t,\xi,.)\|_{L^2(\mathbb{R}^n)} \leqslant \|\hat{f}_0(\xi)\|_{L^2(\mathbb{R}^n)} e^{-c\xi^2 t^3}$$

Key points dissipation N^2 $>>>$ cost $\sqrt{N}$ and spectral inequality's cst N
The semi-group commutes with the frequency cutoff projections.

Drawback of this method

1. **Use of partial Fourier transform** : fails for

$$\partial_t f + v.\nabla_x f + x.\nabla_v f - \Delta_v f = u(t,x,v) 1_\omega(x,v)$$

2. **Cartesian structure :** commutation between the semi-group and the frequency cutoff projections.
3. **Cartesian structure** of control supports
4. **Greedy in spectral analysis :** eigenvalues + eigenfunctions + spectral inequality

We propose a variation of the Lebeau-Robbiano's method, that avoids these restrictions, and allows to extend Le Rousseau and Moyano's result to a large class of equations, with less restrictive control supports

$$\begin{cases} \partial_t f(t,x) - \frac{1}{2}\mathrm{Tr}[Q\nabla_x^2 f(t,x)] - \langle Bx, \nabla_x f(t,x)\rangle = u(t,x)\mathbb{1}_\omega(x)\,, & x \in \mathbb{R}^n \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^n) \end{cases}$$

$Q\,, B \in \mathcal{M}_n(\mathbb{R})$, $Q \geqslant 0$ symmetric and $\mathrm{rank}[Q^{\frac{1}{2}}, BQ^{\frac{1}{2}}, \ldots, B^{n-1}Q^{\frac{1}{2}}] = n$ (hypoellipticity=Kalman).

$$P = \frac{1}{2}\sum_{i,j=1}^{n} q_{i,j}\partial^2_{x_i,x_j} + \sum_{i,j=1}^{n} b_{i,j}x_j\partial_{x_i}$$

Theorem [KB, Pravda-Starov 2016] Let $T > 0$ and ω be an open subset of $\mathbb{R}^n$ such that

$$\exists \delta, r > 0, \forall y \in \mathbb{R}^n\,, \exists y' \in \omega, \quad B(y', r) \subset \omega \text{ and } |y - y'| < \delta\,.$$

Then, the Ornstein-Uhlenbeck equation posed on $L^2(\mathbb{R}^n)$ is null controllable from ω in time $T > 0$.

Examples of Orstein-Uhlenbeck operators

$$\frac{1}{2}\mathrm{Tr}(Q\nabla_x^2) + \langle Bx, \nabla_x \rangle = \frac{1}{2}\sum_{i,j=1}^{n} q_{i,j}\partial^2_{x_i,x_j} + \sum_{i,j=1}^{n} b_{i,j} x_j \partial_{x_i}$$

Application :

1. Heat equation : $Q = I_n$, whatever B
2. $\left(\partial_t + v\partial_x + (ax+bv)\partial_v - \partial_v^2\right) f(t,x,v) = u(t,x,v)\mathbb{1}_\omega(x,v)$
   whatever $(a,b) \in \mathbb{R}^2$: $Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$
   Kolmogorov corresponds to $(a,b) = (0,0)$.
   Using partial Fourier transform wrt x is impossible when $a \neq 0$.

Fourier projections and semigroup do not commute

$$\begin{cases} \partial_t f(t,x) - \frac{1}{2}\mathrm{Tr}[Q\nabla_x^2 f(t,x)] - \langle Bx, \nabla_x f(t,x)\rangle - \frac{1}{2}\mathrm{Tr}(B) f(t,x) = 0\,, \\ f(0,x) = f_0(x). \end{cases}$$

The function h defined by $f(t,x) = h(t, e^{tB}x)e^{\frac{1}{2}\mathrm{Tr}(B)t}$ solves

$$\begin{cases} \partial_t h(t,y) - \frac{1}{2}\mathrm{Tr}[e^{tB}Qe^{tB^T}\nabla_y^2 h(t,y)] = 0\,, \\ h(0,y) = f_0(y)\,. \end{cases}$$

Thus,

$$\widehat{h}(t,\xi) = \widehat{f}_0(\xi)e^{-\frac{1}{2}\int_0^t |Q^{1/2}e^{sB^T}\xi|^2 ds},$$

$$\begin{aligned} \widehat{f}(t,\xi) &= |\det(e^{-tB})|\widehat{h}(t, e^{-tB^T}\xi)e^{\frac{1}{2}\mathrm{Tr}(B)t} \\ &= e^{-\frac{1}{2}\mathrm{Tr}(B)t}\widehat{f}_0(e^{-tB^T}\xi)e^{-\frac{1}{2}\int_0^t |Q^{1/2}e^{(s-t)B^T}\xi|^2 ds}. \qquad (1) \end{aligned}$$

Even if a bounded set of modes could be steered to zero at some time, the passive control phase in the Lebeau-Robbiano method would make them all revive again.

Appropriate dissipation : Gevrey-$\frac{1}{2}$ smoothing

$$\partial_t f(t,x) - \frac{1}{2}\mathrm{Tr}[Q\nabla_x^2 f(t,x)] - \langle Bx, \nabla_x f(t,x)\rangle - \frac{1}{2}\mathrm{Tr}(B) f(t,x) = 0$$

$$\widehat{f}(t,\xi) = e^{-\frac{1}{2}\mathrm{Tr}(B)t}\widehat{f}_0(e^{-tB^T}\xi)e^{-\frac{1}{2}\int_0^t |Q^{1/2}e^{(s-t)B^T}\xi|^2 ds}$$

Lemma : Under Kalman condition, there exists c , t_0 , $k_0 > 0$ such that

$$\forall 0 \leq t \leq t_0, \forall \xi \in \mathbb{R}^n, \quad \int_0^t |Q^{\frac{1}{2}} e^{sB^T}\xi|^2 ds \geq c t^{2k_0+1}|\xi|^2,$$

Let $\pi_k : L^2(\mathbb{R}^n) \to E_k$ the projection onto the closed subspace

$$E_k = \left\{ f \in L^2(\mathbb{R}^n) : \mathrm{supp}(\hat{f}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leq k\} \right\}$$

Then, $\forall T > 0$, $\exists C_T > 1$, $\forall 0 \leq t \leq T$, $\forall k \geq 0$, $\forall g_0 \in L^2(\mathbb{R}^n)$,

$$\|(1-\pi_k)(e^{t\tilde{P}}g_0)\|_{L^2(\mathbb{R}^n)} \leq e^{-\delta(t)k^2}\|g_0\|_{L^2(\mathbb{R}^n)}$$

where $\quad \delta(t) = \frac{1}{C_T}\inf(t,t_0)^{2k_0+1} \geq 0, \quad t \geq 0,$

Rk : $\|g_0\|_{L^2(\mathbb{R}^n)}$ instead of $\|(1-\pi_k)g_0\|_{L^2(\mathbb{R}^n)}$ in the rhs.

Direct proof of the observability inequality by induction

$$\tilde{P} := \frac{1}{2}\mathrm{Tr}(Q\nabla_x^2) + \langle Bx, \nabla_x \rangle + \frac{1}{2}\mathrm{Tr}(B)$$

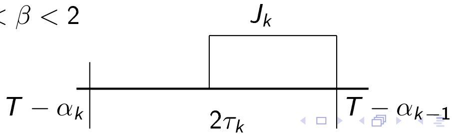
$$\forall T > 0, \exists C_T > 0, \forall g \in L^2(\mathbb{R}^n), \quad \|e^{T\tilde{P}} g\|^2_{L^2(\mathbb{R}^n)} \leq C_T \int_0^T \|e^{t\tilde{P}} g\|^2_{L^2(\omega)} dt$$

Key tools

1. Dissipation : $\|(1-\pi_k)(e^{t\tilde{P}} g_0)\|_{L^2(\mathbb{R}^n)} \leq e^{-\delta(t) k^2} \|g_0\|_{L^2(\mathbb{R}^n)}$
2. Le Rousseau-Moyano spectral inequality : $\exists c_1 > 0$ such that, $\forall N \geq 0$, $\forall g \in L^2(\mathbb{R}^n)$ with $\mathrm{supp}(\hat{g}) \subset B(0, N)$

   $$\|g\|_{L^2(\mathbb{R}^n)} \leqslant c_1 e^{c_1 N} \|g\|_{L^2(\omega)}$$

3. $0 < \rho < \frac{1}{2k_0+1}$, $\quad \tau_k = \frac{K}{4^{k\rho}}$, $\quad \alpha_0 = 0$, $\quad \alpha_k = \sum_{j=1}^k 2\tau_j \underset{k\to\infty}{\longrightarrow} T$, $J_k = [T - \alpha_{k-1} - \tau_k, T - \alpha_{k-1}]$
4. Projections with 2 different scalings : 2^k and $\ell_k := [2^{k\beta}]$ where $1 + \rho(2k_0+1) < \beta < 2$



Second result : hypoelliptic quadratic equations

- **Quadratic operator :** their Weyl symbol is a quadratic form $q : (x, \xi) \in \mathbb{R}^{2n} \mapsto \mathbb{C}$. They are non self-adjoint differential operators, with explicit expression $x^\alpha \xi^\beta \longrightarrow \frac{1}{2}\left(x^\alpha D_x^\beta + D_x^\beta x^\alpha\right)$
- **Hamilton map** and **singular space**

$$F := \frac{1}{2}\begin{pmatrix} \nabla_\xi \nabla_x q & \nabla_\xi^2 q \\ -\nabla_x^2 q & -\nabla_x \nabla_\xi q \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{R})$$

$$S := \Big(\bigcap_{0 \leqslant j \leqslant 2n-1} \mathrm{Ker}\left[\mathrm{Re}\ F (\mathrm{Im}\ F)^j\right] \Big) \cap \mathbb{R}^{2n}$$

We study the class of quadratic operators with $\Re(q) \geqslant 0$ and $S = \{0\}$. They generate contraction semi-groups on $L^2(\mathbb{R}^n)$.

Ex : Kramers-Fokker-Planck equation, with quadratic potential

$$-\Delta_v + \frac{v^2}{4} - \frac{1}{2} + v\partial_x - ax\partial_v \qquad (x, v) \in \mathbb{R}^2$$

Smoothing in the Gelfand-Shilov space $S^{1/2}_{1/2}(\mathbb{R}^n)$

Theorem : [Hitrik, Viola, Pravda-Strarov 2015] Let $q : \mathbb{R}^{2n}_{x,\xi} \to \mathbb{C}$ be a quadratic form with $\mathrm{Re}\, q \geq 0$ and $S = \{0\}$. Then, there exist $C_0\,, t_0 > 0$ such that for all $t \in (0, t_0)$, $u \in L^2(\mathbb{R}^n)$, $\alpha, \beta \in \mathbb{N}^n$

- $\left\| x^\alpha \partial_x^\beta (e^{-tq^w} u) \right\|_{L^\infty(\mathbb{R}^n)} \leqslant \frac{C_0^{1+|\alpha|+|\beta|}}{t^{\frac{2k_0+1}{2}(|\alpha|+|\beta|+2n+s)}} (\alpha!)^{1/2} (\beta!)^{1/2} \|u\|_{L^2(\mathbb{R}^n)}$
- $\left\| e^{\delta(t)(D_x^2+x^2)} e^{-tq^w} \right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C_0$ where $\delta(t) := \frac{\inf(t,t_0)^{2k_0+1}}{C_0}$

Corollary : $\left\| (1-\pi_k)(e^{-tq^w} u) \right\|_{L^2(\mathbb{R}^n)} \leq C_0 e^{-k\delta(t)} \|u\|_{L^2(\mathbb{R}^n)}$
where π_k is the orthogonal projection onto the energy levels lower than k associated to the Harmonic oscillator on $\mathbb{R}^n$

$$\pi_k u := \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\leq k}} (u, \psi_\alpha)\psi_\alpha \qquad \left(D_x^2 + x^2\right)\psi_\alpha = (2|\alpha| + n)\psi_\alpha$$

Theorem : [B, Pravda-Starov 2015] Let $q : \mathbb{R}^{2n}_{x,\xi} \to \mathbb{C}$ be a quadratic form with $\mathrm{Re}\, q \geq 0$, $S = \{0\}$ and P be the associated differential operator. Let $T > 0$ and ω be an open subset of $\mathbb{R}^n$ that satisfies

$$\exists \delta\,, r > 0 \,/\, \forall y \in \mathbb{R}^d\,, \exists y' \in \omega \,/\, B(y', r) \subset \omega \text{ and } |y - y'| < \delta\,. \qquad (2)$$

Then, for every $f_0 \in L^2(\mathbb{R}^n)$, there exists $u \in L^2((0,T) \times \omega)$ such that the solution of

$$\begin{cases} (\partial_t + P) f(t,x) = u(t,x) 1_\omega(x)\,, & x \in \mathbb{R}^n\,, \\ f(0,x) = f_0\,, & x \in \mathbb{R}^n\,, \end{cases}$$

satisfies $f(T,.) = 0$.

Proof

- **Spectral inequality for Hermite functions :** There exists $C_1 > 0$ such that for every $k \in \mathbb{N}^*$ and $(b_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{C}^{\mathbb{N}^n}$,

$$\left(\int_{\mathbb{R}^n} \Big| \sum_{|\alpha| \leqslant k} b_\alpha \psi_\alpha(x) \Big|^2 dx \right)^{1/2} \leqslant C_1 e^{C_1 \sqrt{k}} \left(\int_{\omega} \Big| \sum_{|\alpha| \leqslant k} b_\alpha \psi_\alpha(x) \Big|^2 dx \right)^{1/2} .$$

  The proof relies on a global Carleman elliptic estimate for $D_s^2 + D_x^2 + x^2$ $\sim$ [Le Rousseau-Moyano 2015].

- **Lebeau-Robbiano's method :** direct approach for observability (induction).

$$\left\| (1 - \pi_k)(e^{-tq^w} u) \right\|_{L^2(\mathbb{R}^n)} \leq C_0 e^{-k\delta(t)} \|u\|_{L^2(\mathbb{R}^n)}$$

- **Kramers-Fokker-Planck** : $-\Delta_v + \frac{v^2}{4} + v\partial_x - ax\partial_v \rightarrow S = \{0\}$
- **Ornstein-Uhlenbeck in weighted spaces** : $Q, B \in \mathcal{M}_n(\mathbb{R})$, Q symmetric $\geqslant 0$

$$P := \frac{1}{2}\mathrm{Tr}(Q\nabla_x^2) + \langle Bx, \nabla_x\rangle \quad \text{with} \quad \mathrm{rk}\left[\sqrt{Q}, B\sqrt{Q}, ..., B^{n-1}\sqrt{Q}\right] = n$$

$S \neq \{0\}$. But when $\Re[\mathrm{Sp}(B)] \subset (-\infty, 0)$ (CNS), there exists a gaussian invariant measure $d\mu(x) = \rho(x)dx$ and we may associate to the operator P acting on $L^2(\mathbb{R}^n, d\mu)$, the quadratic operator $\mathcal{L}$ acting on $L^2(\mathbb{R}^n, dx)$,

$$\mathcal{L}u = -\sqrt{\rho}P\big((\sqrt{\rho})^{-1}u\big) - \frac{1}{2}\mathrm{Tr}(B)u.$$

Then $\mathcal{L} = q^w(x, D_x)$ is quadratic with $\Re q \geq 0$ and $S = \{0\}$.

- **Ex** : $\Delta_v + v\nabla_x - (x+v)\nabla_v \quad$ in $L^2(e^{-(x^2+v^2)}dxdv)$

Conclusion, perspectives, open problems

- **Done :** Non autonomous Ornstein-Uhlenbeck operators

$$\partial_t f(t,x) - \frac{1}{2}\mathrm{Tr}[A(t)^T A(t) \nabla_x^2 f(t,x)] - \langle B(t)x, \nabla_x f(t,x)\rangle = u(t,x)\mathbb{1}_\omega(x)$$

under generalized Kalman condition

$$\exists \bar{t} \in [0,T], p \in \mathbb{N} \text{ such that } \mathrm{rk}[\tilde{A}_0(\bar{t}) \quad \dots \quad \tilde{A}_p(t)] = n$$

$$\tilde{A}_0(t) := B(t)\,, \quad \tilde{A}_{k+1}(t) := \tilde{A}_k'(t) - B(t)\tilde{A}_k(t)\,, \forall k \geqslant 0$$

- **Under progress :** Ornstein-Uhlenbeck operators without Kalman, quadratic operators with $S = \{0\}$, semilinear equations
- **Long term :** fully nonlinear models
- **Conclusion :** Null controllability = smoothing

Miscellaneous facts about quadratic operators

The **maximal closed realization** of a **quadratic operator** $q^w(x, D_x)$ with **domain**

$$D(q) = \{u \in L^2(\mathbb{R}^n) : q^w(x, D_x)u \in L^2(\mathbb{R}^n)\}$$

coincides with the **graph closure** of its restriction to the **Schwartz space**. The **adjoint operator** $q^w(x, D_x)^*$ is given by $\overline{q}^w(x, D_x)$ with domain $D(\overline{q})$

When $\mathrm{Re}\, q \geq 0$, the operator $q^w(x, D_x)$ is **maximally accretive**

$$\forall u \in D(q), \quad \mathrm{Re}(q^w(x, D_x)u, u)_{L^2} \geq 0$$

and generates a **contraction semigroup** $(e^{-tq^w})_{t \geq 0}$ on $L^2(\mathbb{R}^n)$

The **Hamilton map** $F \in M_{2n}(\mathbb{C})$ of the **quadratic operator** $q^w(x, D_x)$ is uniquely defined by the identity

$$q(X; Y) = \langle QX, Y \rangle = \sigma(X, FY), \qquad X, Y \in \mathbb{R}^{2n}$$

where σ is the **canonical symplectic form**, that is,

$$F = \sigma Q = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} Q$$

Singular space

The **singular space** of a quadratic operator $q^w(x, D_x)$ is defined as the following **finite intersection** of kernels (**algebraic definition**)

$$S = \Big(\bigcap_{j=0}^{2n-1} \mathrm{Ker}\big[\mathrm{Re}\ F(\mathrm{Im}\ F)^j\big]\Big) \cap \mathbb{R}^{2n} \subset \mathbb{R}^{2n}$$

where F denotes the **Hamilton map** of its Weyl symbol q *[Hitrik, Pravda-Starov]*

When $\mathrm{Re}\ q \geq 0$, the **singular space** is the subset in the phase space where all the **iterated Poisson brackets** $H^k_{\mathrm{Im}q}\mathrm{Re}\ q$ vanish (**dynamical definition**)

$$S = \big\{X \in \mathbb{R}^{2n} : H^k_{\mathrm{Im}q}\mathrm{Re}\ q(X) = 0,\ k \geq 0\big\}$$

with

$$H_{\mathrm{Im}q} = \frac{\partial \mathrm{Im}\ q}{\partial \xi} \cdot \frac{\partial}{\partial x} - \frac{\partial \mathrm{Im}\ q}{\partial x} \cdot \frac{\partial}{\partial \xi}$$

The **singular space** corresponds to the **subset** of points $X_0 \in \mathbb{R}^{2n}$ where the function

$$t \mapsto \mathrm{Re}\ q(e^{tH_{\mathrm{Im}q}}X_0)$$

vanishes at an **infinite order** at $t = 0$, that is, is identically equal to **zero**.

Accretive quadratic operators with zero singular space

Let $q^w(x, D_x)$ be a **quadratic operator** whose Weyl symbol has a **non-negative real part** $\mathrm{Re}\ q \geq 0$ and a **zero singular space** $S = \{0\}$. Then,

$$\forall T > 0, \quad \langle \mathrm{Re}\ q \rangle_T(X) = \frac{1}{2T} \int_{-T}^{T} (\mathrm{Re}\ q)(e^{tH_{\mathrm{Im}q}} X) dt \gg 0$$

The **contraction semigroup** $(e^{-tq^w})_{t \geq 0}$ on $L^2(\mathbb{R}^n)$ is **smoothing** in the **Gelfand-Shilov space** $S_{1/2}^{1/2}(\mathbb{R}^n)$ for **any positive time** $t > 0$

$$\forall u \in L^2(\mathbb{R}^n), \forall t > 0, \quad e^{-tq^w} u \in S_{1/2}^{1/2}(\mathbb{R}^n)$$

The **Gelfand-Shilov spaces** $S_\nu^\mu(\mathbb{R}^n)$ with $\mu, \nu > 0$, $\mu + \nu \geq 1$, are the spaces of functions $f \in C^\infty(\mathbb{R}^n)$ s.t.

$$\exists C > 1, \forall \alpha, \beta \in \mathbb{N}^n, \quad \sup_{x \in \mathbb{R}^n} |x^\beta \partial_x^\alpha f(x)| \leq C^{1+|\alpha|+|\beta|} (\alpha!)^\mu (\beta!)^\nu$$

Symmetric Gelfand-Shilov spaces $S_\mu^\mu(\mathbb{R}^n)$ with $\mu \geq 1/2$:

$$f \in S_\mu^\mu(\mathbb{R}^n) \Leftrightarrow f \in L^2(\mathbb{R}^n), \ \exists t_0 > 0, \ \|e^{t_0 \mathcal{H}^{\frac{1}{2\mu}}} f\|_{L^2(\mathbb{R}^n)} < +\infty$$

where $\mathcal{H} = -\Delta_x + |x|^2$ is the **harmonic oscillator**