State estimation for linear age-structured population diffusion models

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The problem in a glance

A classical model for age-space structured populations is given by

\[
\begin{aligned}
\frac{\partial p}{\partial t}(a, x, t) + \frac{\partial p}{\partial a}(a, x, t) \\
= -\mu(a)p(a, x, t) + k\Delta p(a, x, t),
\end{aligned}
\]

\( a \in (0, a^*), \ x \in \Omega, \ t > 0, \)

\[ p(a, x, t) = 0, \quad a \in (0, a^*), \ x \in \partial\Omega, \ t > 0, \]

\[ p(a, x, 0) = p_0(a, x), \quad a \in (0, a^*), \ x \in \Omega, \]

\[ p(0, x, t) = \int_0^{a^*} \beta(a)p(a, t, x)\,da, \quad x \in \Omega, \ t > 0. \]

- \( p(a, x, t) \): distribution density of the population of age \( a \) at spatial position \( x \) at time \( t \);
- \( a^* \): maximal life expectancy;
- \( k \): diffusion coefficient;
- \( \mu(a), \beta(a) \): death and birth rates (independent of \( x \)).
The problem in a glance

Figure: Typical birth and death rates.
The problem in a glance

Estimation problem
Knowing the output $y(t) := p|_{(a_1, a_2) \times O}$ (but assuming that $p_0$ is unknown), estimate $p(a, x, T)$ for all $a \in (0, a^*)$ and $x \in \Omega$, as $T \to +\infty$. 
The problem in a glance

\[
\begin{cases}
\dot{p}(t) = Ap(t), & t \in (0, T) \\
p(0) = p_0, \\
y(t) = Cp(t), & t \in (0, T),
\end{cases}
\]

where \( C \in \mathcal{L}(X, Y) \), \( Y := L^2((a_1, a_2) \times \mathcal{O}) \) is defined by

\[ C\varphi := \varphi|_{(a_1, a_2) \times \mathcal{O}} \text{ for all } \varphi \in X. \]

We introduce the Luenberger observer

\[
\begin{cases}
\dot{\hat{p}}(t) = A\hat{p}(t) + L(C\hat{p}(t) - y(t)), & t \in (0, T) \\
\hat{p}(0) = 0,
\end{cases}
\]

where \( L \in \mathcal{L}(Y, X) \) is a linear operator to be defined.

Then the error \( e := \hat{p} - p \) satisfies

\[
\begin{cases}
\dot{e}(t) = (A + LC)e(t), & t \in (0, T) \\
e(0) = -p_0.
\end{cases}
\]
The problem in a glance

Goal

Find $L$ such that $e^{t(A+LC)}$ exponentially stable (detectability).
The problem in a glance

Goal

Find $L$ such that $e^{t(A+LC)}$ exponentially stable (detectability).

How?

Spectrum of $A$: an infinite number of stable modes and a finite number of unstable modes.

Design an infinite dimensional Luenberger observer via a finite dimensional stabilizing operator.
Selective bibliography

**Population Dynamics**
- **Semigroup properties:** Song et al., Chan, Guo, Li et al., Langlais, Walker
- **Controllability problems:** Ainseba, Anita, Iannelli, Langlais, Echarroudi, Maniar, Traoré, Kavian
- **Inverse problems:** Traoré, Rundell, Di Blasio, Lorenzi, Perasso, Picart
- **Numerical aspects:** Lopez, Trigiante, Milner, Kim, Huyer, Ayati, Dupont, Pelovska, Gerardo-Giorda

**State Space Splitting**
- **Abstract setting:** Russel, Triggiani, Jacobson & Nett, Jacob & Zwart
- **Stabilization of PDE:** Barbu & Triggiani, Raymond et al., Badra & Takahashi
Outline

1. Spectral properties of the operator
2. Detectability
3. Application: observer design for populations dynamics
4. Numerical results
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1. Spectral properties of the operator
2. Detectability
3. Application: observer design for populations dynamics
4. Numerical results
The model

\[
\frac{\partial p}{\partial t}(a, x, t) = -\frac{\partial p}{\partial a}(a, x, t) - \mu(a)p(a, x, t) + k\Delta p(a, x, t), \quad a \in (0, a^*), \ x \in \Omega, \ t > 0,
\]

\[
p(a, x, t) = 0, \quad a \in (0, a^*), \ x \in \partial\Omega, \ t > 0,
\]

\[
p(a, x, 0) = p_0(a, x), \quad a \in (0, a^*), \ x \in \Omega,
\]

\[
p(0, x, t) = \int_0^{a^*} \beta(a)p(a, t, x) \, da, \quad x \in \Omega, \ t > 0.
\]
Assumptions

Typical assumptions on the birth and death rates $\beta$ and $\mu$:

1. $\beta \in L^\infty(0,a^*)$, $\beta \geq 0$ a.e. in $(0,a^*)$;
2. $\mu \in L^1_{\text{loc}}(0,a^*)$, $\mu \geq 0$ a.e. in $(0,a^*)$ and

$$\lim_{a \to a^*} \int_0^a \mu(s) \, ds = +\infty.$$ 

We also introduce the function

$$\Pi(a) := \exp \left( - \int_0^a \mu(s) \, ds \right)$$

which represents the probability to survive at age $a > 0$. In particular

$$\lim_{a \to a^*} \Pi(a) = 0.$$
We introduce the Hilbert space $X := L^2 ((0, a^*) \times \Omega)$ and let $A$ be defined by:

$$
D(A) = \left\{ \varphi \in X \cap L^2 ((0, a^*), H^1_0(\Omega)) \mid - \frac{\partial \varphi}{\partial a} - \mu \varphi + k \Delta \varphi \in X; \right. \\
\varphi(a, \cdot)|_{\partial \Omega} = 0 \text{ for almost all } a \in (0, a^*); \\
\varphi(0, x) = \int_0^{a^*} \beta(a) \varphi(a, x) \, da \text{ for almost all } x \in \Omega \right\}
$$

$$
A \varphi = - \frac{\partial \varphi}{\partial a} - \mu \varphi + k \Delta \varphi, \quad \forall \varphi \in D(A).
$$
The population dynamics problem reads then

\[ \begin{cases} \dot{p}(t) = Ap(t), & t > 0 \\ p(0) = p_0. \end{cases} \]

**Theorem (Chan and Guo, 1989)**

- *A is the infinitesimal generator of a $C_0$–semigroup $e^{tA}$ on $X$.*
- *If $p_0 \in X$, there exists a unique solution $p \in C([0, \infty), X)$.*
- *If $p_0 \in \mathcal{D}(A)$, there exists a unique solution $p \in C([0, \infty), \mathcal{D}(A)) \cap C^1([0, \infty), X)$.***
McKendrick–Von Foerster model (1959) describes the diffusion free case ($k = 0$):

\[
\begin{align*}
\frac{\partial p}{\partial t}(a,t) &= -\frac{\partial p}{\partial a}(a,t) - \mu(a)p(a,t), \quad a \in (0, a^*), \ t > 0, \\
p(a,0) &= p_0(a), \quad a \in (0, a^*), \\
p(0,t) &= \int_0^{a^*} \beta(a)p(a,t) \, da, \quad t > 0.
\end{align*}
\]
The population operator $A_0$ corresponding to the above system is defined as follows

$$\mathcal{D}(A_0) = \left\{ \varphi \in L^2(0, a^*) \mid -\frac{d\varphi}{da} - \mu \varphi \in L^2(0, a^*); \varphi(0) = \int_0^{a^*} \beta(a) \varphi(a) \, da \right\}.$$ 

Then $A_0 \varphi = -\frac{d\varphi}{da} - \mu \varphi$, $\forall \varphi \in \mathcal{D}(A_0)$.

Then the McKendrick–Von Foerster model reads then

$$\begin{cases}
\dot{p}(t) = A_0 p(t), & t > 0 \\
p(0) = p_0.
\end{cases}$$
Theorem (Song et al., 1982)

1. $A_0$ has compact resolvent and its spectrum is constituted of a countable (infinite) set of isolated eigenvalues with finite algebraic multiplicity.

2. The eigenvalues $(\lambda^0_n)_{n \geq 1}$ of $A_0$ (counted without multiplicity) are the (complex) solutions of the characteristic equation

$$F(\lambda) := \int_0^{a^*} \beta(a)\Pi(a)e^{-\lambda a} \, da = 1.$$

3. The eigenvalues $(\lambda^0_n)_{n \geq 1}$ are of geometric multiplicity one:

$$\varphi^0_n(a) = e^{-\lambda^0_n a}\Pi(a) = e^{-\lambda^0_n a - \int_0^a \mu(s) \, ds}.$$

4. Every vertical strip of the complex plane contains a finite number of eigenvalues of $A_0$. 

Diffusion free model

Theorem (Song et al., 1982)

The operator $A_0$ has a unique real eigenvalue $\lambda^0_1$. Moreover:

1. $\lambda^0_1$ is of algebraic multiplicity one;
2. $\lambda^0_1 > 0$ ($< 0$) $\iff$ $F(0) = \int_0^{a^*} \beta(a) \Pi(a) \, da > 1$ ($< 1$);
3. $\lambda^0_1$ is a real dominant eigenvalue:

$$\lambda^0_1 > \text{Re} (\lambda^n_0), \quad \forall n \geq 2.$$
Back to the problem with diffusion

\[ D(A) = \left\{ \varphi \in X \cap L^2((0, a^*), H^1_0(\Omega)) \left| -\frac{\partial \varphi}{\partial a} - \mu \varphi + k \Delta \varphi \in X; \varphi(a, \cdot)|_{\partial \Omega} = 0 \text{ for almost all } a \in (0, a^*); \varphi(0, x) = \int_0^{a^*} \beta(a) \varphi(a, x) \, da \text{ for almost all } x \in \Omega \right. \right\} \]

\[ A \varphi = -\partial_a \varphi - \mu \varphi + k \Delta \varphi, \quad \forall \varphi \in D(A). \]

Let \( 0 < \lambda^D_1 < \lambda^D_2 \leq \lambda^D_3 \leq \cdots \) be the increasing sequence of eigenvalues of \(-k \Delta\) with Dirichlet boundary conditions and let \((\varphi^D_n)_{n \geq 1}\) be a corresponding orthonormal basis of \(L^2(\Omega)\).
Spectral properties

Theorem (Chan and Guo, 1989)

1. \( A \) has compact resolvent and its (pure point) spectrum is

\[
\sigma(A) = \{ \lambda_i^0 - \lambda_j^D | i, j \in \mathbb{N}^* \}
\]

2. The eigenspace associated to an eigenvalue \( \lambda \) of \( A \) is given by

\[
\text{Span} \left\{ \varphi_i^0(a) \varphi_j^D(x) = e^{-\lambda_i^0 a \Pi(a)} \varphi_j^D(x) \bigg| \lambda_i^0 - \lambda_j^D = \lambda \right\}.
\]

3. The real eigenvalue \( \lambda_1 \) of \( A \) is dominant:

\[
\lambda_1 = \lambda_1^0 - \lambda_1^D > \text{Re}(\lambda), \quad \forall \lambda \in \sigma(A), \; \lambda \neq \lambda_1.
\]

4. \( \lambda_1 \) is a simple eigenvalue, the corresponding eigenspace being generated by

\[
\varphi_1(a, x) := \varphi_1^0(a) \varphi_1^D(x) = e^{-\lambda_1^0 a \Pi(a)} \varphi_1^D(x).
\]
In this example, there is only 1 unstable eigenvalue $\lambda_1$:

\[
\Re (\lambda_n^0) < \lambda_1^D < \lambda_1^0 < \lambda_2^D, \quad \forall n \geq 2
\]

\[\Rightarrow \lambda_1 = \lambda_1^0 - \lambda_1^D > 0, \quad \Re (\lambda_n) < 0, \quad \forall n \geq 2\]
Compactness & Stability

Proposition (Chan and Guo, 1989)

The semigroup \( e^{tA} \) generated on \( X \) by \( A \) is compact for \( t \geq a^* \).

This implies in particular that (see Zabczyk, 1975)

\[
\omega_a(A) = \omega_0(A)
\]

where \( \omega_a(A) := \lim_{t \to +\infty} t^{-1} \ln \|e^{tA}\| \) denotes the growth bound of \( e^{tA} \) and \( \omega_0(A) := \sup \{ \text{Re} \lambda \mid \lambda \in \sigma(A) \} \) the spectral bound of \( A \).

Consequence

The above condition ensures that the exponential stability of \( e^{tA} \) is equivalent to the condition

\[
\omega_0(A) = \sup \{ \text{Re} \lambda \mid \lambda \in \sigma(A) \} < 0.
\]
Outline

1. Spectral properties of the operator
2. Detectability
3. Application: observer design for populations dynamics
4. Numerical results
Consider

- $A : D(A) \to X$ with compact resolvent on a Hilbert space $X$ generating a $C_0$-semigroup in $X$,
- $C \in \mathcal{L}(X, Y)$, where $Y$ is another Hilbert space.

We assume

(A1) $A$ admits $M$ eigenvalues (counted without multiplicities) with real part greater or equal than 0:

$$\cdots \leq \Re \lambda_{M+2} \leq \Re \lambda_{M+1} < 0 \leq \Re \lambda_M \leq \cdots \leq \Re \lambda_2 \leq \Re \lambda_1.$$

(A2) We have the equality

$$\omega_a(A) = \omega_0(A).$$
Detectability

Definition

The pair \((A, C)\) is detectable if there exists \(L \in \mathcal{L}(Y, X)\) such that \((A + LC)\) generates an exponentially stable semigroup.

We are going to show that:

\[
\begin{align*}
\text{Spectral observability of unstable eigenfunctions of } A &
\Rightarrow A\varphi = \lambda \varphi \text{ for } \lambda \in \Sigma_+ \text{ and } C\varphi = 0 \\
&\Rightarrow \varphi = 0
\end{align*}
\]

\[
\Downarrow
\]

Detectability of the finite dimensional system \((A^+, C^+)\)

\[
\Downarrow
\]

Detectability of the infinite dimensional system \((A, C)\)
We set $\Sigma_+ := \{\lambda_1, \ldots, \lambda_M\}$ and let $\Gamma_+$ be a positively oriented curve enclosing $\Sigma_+$ but no other point of the spectrum of $A$. Let $P_+ : X \to X$ be the projection operator defined by

$$P_+ := -\frac{1}{2\pi i} \int_{\Gamma_+} (\xi - A)^{-1} d\xi.$$
We set $X_+ := P_+X$ and $X_- := (I - P_+)X$, and then $P_+$ provides the following decomposition of $X$:

$$X = X_+ \oplus X_-.$$  

Following Russell and Triggiani, we can decompose our system into two subsystems:

- a finite dimensional system to be stabilized,
- a stable infinite dimensional system.

More precisely, $X_+$ and $X_-$ are invariant subspaces under $A$ (since $A$ and $P_+$ commute) and the spectra of the restricted operators $A \mid_{X_+}$ and $A \mid_{X_-}$ are respectively $\Sigma_+$ and $\Sigma_- := \sigma(A) \setminus \Sigma_+$. We also define:

$$A_+ := A\mid_{\mathcal{D}(A) \cap X_+} : \mathcal{D}(A) \cap X_+ \to X_+, \quad A_- := A\mid_{\mathcal{D}(A) \cap X_-} : \mathcal{D}(A) \cap X_- \to X_-.$$
If $A$ is **diagonalizable**, the space $X_+ = P_+X$ is the **finite dimensional** space spanned by the eigenfunctions of $A$ associated to the **unstable eigenvalues**:

$$X_+ = \bigoplus_{k=1}^{M} \ker(A - \lambda_k).$$

and

$$\dim X_+ = \sum_{k=1}^{M} m^G_k.$$

where $m^G_k := \dim \ker(A - \lambda_k)$ is the **geometric multiplicity** of $\lambda_k$. 
In the general case, the space $X_+$ is the finite dimensional space spanned by the generalized eigenfunctions of $A$ associated to the unstable eigenvalues:

$$X_+ = \bigoplus_{k=1}^{M} \text{Ker} \ (A - \lambda_k)^{m_k^P}$$

where $m_k^P$ is the multiplicity of the pole $\lambda_k$ in the resolvent $(A - \lambda)^{-1}$.

The space $\text{Ker} \ (A - \lambda_k)^{m_k^P}$ is called the generalized eigenspace associated to $\lambda_k$. Its dimension $m_k^A$ is the algebraic multiplicity of $\lambda_k$.

$$\text{dim} \ X_+ = \sum_{k=1}^{M} m_k^A.$$
Theorem

Let

- $Q_+ : Y \to Y_+ := CX_+$ be the orthogonal projection operator from $Y$ to $Y_+$,
- $i_{X_+} : X_+ \to X$ be the embedding operator from $X_+$ into $X$.

Set

$$C_+ = Ci_{X_+} \in \mathcal{L}(X_+, Y_+)$$

and assume that the finite dimensional projected system $(A_+, C_+)$ is detectable through $L_+ \in \mathcal{L}(Y_+, X_+)$. Then, the infinite dimensional system $(A, C)$ is detectable through

$$L = i_{X_+} L_+ Q_+ \in \mathcal{L}(Y, X).$$
Proof

For $L \in \mathcal{L}(Y, X)$, consider the system

$$\dot{z}(t) = (A + LC)z(t).$$

If we write $z = z_+ + z_-$ where $z_+ := P_+z$ and $z_- := (I - P_+)z$, by applying $P_+$ and $(I - P_+)$ to the above equation, we obtain a corresponding splitting of the system into two subsystems:

$$\begin{align*}
\dot{z}_+(t) &= A_+z_+(t) + P_+LCz(t), \\
\dot{z}_-(t) &= A_-z_-(t) + (I - P_+)LCz(t).
\end{align*}$$

Taking $L = i_{X_+}L_+Q_+$ and using the identities $P_+i_{X_+} = \operatorname{Id}_{X_+}$ and $(I - P_+)i_{X_+} = 0$, we obtain

$$\begin{align*}
\dot{z}_+(t) &= A_+z_+(t) + L_+Q_+Cz(t), \\
\dot{z}_-(t) &= A_-z_-(t).
\end{align*}$$
Proof

It follows from assumption \((A2)\) that \(z_-\) is exponentially stable:

\[
\|z_-(t)\| \leq Ke^{-\omega_- t} \|z_-(0)\|
\]

where \(0 < \omega_- \leq -\text{Re} \lambda_{M+1}\). On the other hand, by using \(C_+ = Q_+ Ci_{X_+}\) and since \(i_{X_+}z_+ = z_+\), we have

\[
\begin{align*}
\dot{z}_+(t) &= A_+z_+(t) + L_+Q_+C(z_+(t) + z_-(t)) \\
&= A_+z_+(t) + L_+Q_+Ci_{X_+}z_+(t) + L_+Q_+Cz_-(t) \\
&= (A_+ + L_+C_+)z_+(t) + L_+Q_+Cz_-(t).
\end{align*}
\]
Proof

Using Duhamel's formula, we get

\[ z_+(t) = T^+_t z_+(0) + \int_0^t T^+_{t-s} L_+ Q_+ C z_-(s) ds, \]

where \( T^+_t \) is the semigroup generated by \( A_+ + L_+ C_+ \), which is exponentially stable by the detectability assumption, i.e. there exists \( \omega_+ > 0 \) such that

\[ \| T^+_t x \| \leq K e^{-\omega_+ t} \| x \| \quad \forall x \in X_+, \forall t > 0. \]

Combined with exponential stability of \( z_- \), this yields

\[ \| z_+(t) \| \leq K \left\{ e^{-\omega_+ t} \| z_+(0) \| + \| L_+ \| \| C \| \int_0^t e^{-\omega_+(t-s)} e^{-\omega_- s} \| z_-(0) \| ds \right\}, \]

and consequently

\[ \| z_+(t) \| \leq K \left( e^{-\omega_+ t} + \| L_+ \| \| C \| \frac{e^{-\omega_+ t} - e^{-\omega_- t}}{\omega_- - \omega_+} \right) \| z_0 \|. \]
Proof

It is then sufficient to choose $\omega_+$ small enough such that $0 < \omega_+ < \omega_-$ to have the exponential decay of $t \mapsto z_+(t)$:

$$\|z_+(t)\| \leq K e^{-\omega_+ t} \|z_0\|, \quad t > 0.$$  

We have thus proved the exponential decay of $z = z_+ + z_-$.  

$\blacksquare$
The following result provide a sufficient condition of Hautus type for the detectability of the finite dimensional projected system \((A_+, C_+).\)

**Proposition**

*If the spectral observability condition (Hautus test)*

\[
(A\varphi = \lambda \varphi \text{ for } \lambda \in \Sigma_+ \text{ and } C\varphi = 0) \implies \varphi = 0
\]

*is satisfied, then \((A_+, C_+)\) is detectable.*

**Proof:** Since \(C_+ z_+ = C z_+\) for any \(z_+ \in X_+\), if the Hautus test is satisfied, then it is clear that the following Hautus test is also satisfied:

\[
(\varphi \in \mathcal{D}(A) \cap X_+ \mid A_+ \varphi = \lambda \varphi \text{ and } C_+ \varphi = 0) \implies \varphi = 0.
\]

As the above system is finite dimensional, \((A_+, C_+)\) is detectable. ■
Corollaire

If the Hautus test is satisfied, then \((A, C')\) is detectable via the stabilizing output injection operator \(L\) defined previously.

- The matrices \(A_+\) and \(C_+\) are in practice of small size: their dimensions are respectively \(\dim X_+ \times \dim X_+\) and \(\dim Y_+ \times \dim X_+\).
- The stabilizing operator \(L_+\) of the finite dimensional system \((A_+, C_+)\) can be determined by solving a finite dimensional algebraic Riccati equation.
Outline

1. Spectral properties of the operator
2. Detectability
3. Application: observer design for populations dynamics
4. Numerical results
Assumptions \((A1)\) \((M < \infty \text{ unstable eigenvalues})\) and \((A2)\) 
\((\omega_a(A) = \omega_0(A))\) are satisfied for our population model and the problem of determining the stabilizing operator \(L\) for \((A, C')\) fits into the framework described above.
It only remains to verify that the Hautus test is satisfied for our system \((A, C')\):

**Lemma**

If \(\varphi \in \mathcal{D}(A)\) satisfies \(A\varphi = \lambda \varphi\) for \(\lambda \in \Sigma_+\) and \(C\varphi = 0\), then \(\varphi\) vanishes identically.
Proof of the spectral observability

Let $\lambda$ be an unstable eigenvalue of $A$ and let $\varphi \in \mathcal{D}(A)$ satisfying $A\varphi = \lambda \varphi$. Decomposing $\varphi(0, x)$ in the basis of $L^2(\Omega)$ constituted of the eigenfunctions of $-k\Delta$, the unique solution of the evolution system

$$
\begin{cases}
\frac{\partial \varphi}{\partial a}(a, x) = k\Delta \varphi(a, x) - (\lambda + \mu)\varphi(a, x), & a \in (0, a^*), \ x \in \Omega, \\
\varphi(a, x) = 0, & a \in (0, a^*), \ x \in \partial \Omega, \\
\varphi(0, x) = \sum_{j \in \mathbb{N}} \alpha_j \varphi^D_j(x), & x \in \Omega,
\end{cases}
$$

is given by

$$\varphi(a, x) = \sum_{j \in \mathbb{N}} \alpha_j e^{-(\lambda + \lambda^D_j)a} \Pi(a) \varphi^D_j(x).$$

Plugging the above expression in the renewal equation, we obtain

$$
\sum_{j \in \mathbb{N}} \alpha_j \varphi^D_j(x) = \sum_{j \in \mathbb{N}} \alpha_j \left( \int_0^{a^*} \beta(a) e^{-(\lambda + \lambda^D_j)a} \Pi(a) \right) \varphi^D_j(x).
$$

We see that is equivalent to, for any $j \in \mathbb{N}$, either $\alpha_j = 0$, either $\lambda + \lambda^D_j$ solves the characteristic equation of the diffusion free problem.
Consequently, we have

\[ \varphi(a, x) = \sum_{j, \lambda + \lambda_j^D \in \sigma(A_0)} \alpha_j e^{-(\lambda + \lambda_j^D)a} \Pi(a) \varphi_j^D(x). \]

The condition \( C\varphi = 0 \) reads then

\[ \sum_{j, \lambda + \lambda_j^D \in \sigma(A_0)} \alpha_j e^{-(\lambda + \lambda_j^D)a} \varphi_j^D |_{\partial(x)} = 0, \quad a \in (a_1, a_2). \]

Since the eigenfunctions of \(-k\Delta\) with Dirichlet boundary conditions are analytic, we immediately obtain that \( \varphi = 0 \). \( \blacksquare \)
Theorem

Let \( p_0 \in X \) and assume that \( y(t) = p|_{(a_1, a_2) \times \mathcal{O}} \ (t > 0) \) is known. Let \( \hat{p} \) the observer defined by

\[
\begin{align*}
\dot{\hat{p}}(t) &= A\hat{p}(t) + L(C\hat{p}(t) - y(t)), \quad t \in (0, T) \\
\hat{p}(0) &= 0,
\end{align*}
\]

where \( L \in \mathcal{L}(Y, X) \) is the stabilizing operator defined by

\[
L = i_{X^+}L + Q_+ \in \mathcal{L}(Y, X).
\]

Then, there exist \( M, \omega > 0 \) such that

\[
\|\hat{p}(t) - p(t)\| \leq Me^{-\omega t} \|p_0\|, \quad t > 0.
\]
1 Spectral properties of the operator
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Taking $\Omega = (0, \pi)$ and assuming that $p_0$ is an unknown initial data, we want to estimate $p$ at time $t = T$ where $p$ solves:

$$
\begin{align*}
\frac{\partial}{\partial t} p(a, x, t) + \frac{\partial}{\partial a} p(a, x, t) &= -\mu(a) p(a, x, t) + \frac{\partial}{\partial x} p(a, x, t), \\
&\quad a \in (0, a^*), x \in (0, \pi), t > 0,
\end{align*}
$$

$$
\begin{align*}
p(a, 0, t) &= p(a, \pi, t) = 0, \\
p(a, x, 0) &= p_0(a, x), \\
p(0, x, t) &= \int_0^{a^*} \beta(a) p(a, x, t) \, da,
\end{align*}
$$

provided we know the observation

$$
y(t) = p(t)_{|(0,a^*) \times (\pi/3,2\pi/3)}, \quad t \in (0, T).
$$
Taking $a^* = 2$, we choose the fertility and mortality function to be

$$\beta(a) = 10 a (a^* - a) \exp \left\{ -20(a - a^*/3)^2 \right\}, \quad \mu(a) = (a^* - a)^{-1}.$$  

Note that the function $\Pi(a)$ can be computed explicitly:

$$\Pi(a) = \exp \left( - \int_0^a \mu(s) \, ds \right) = \frac{a^* - a}{a^*}.$$  

**Figure**: The fertility and mortality functions.
First test: initial state = unstable eigenfunction

Under these assumptions, there is a unique unstable eigenvalue \( \lambda_1 = \lambda_1^0 - \pi^2 \) (where \( \lambda_1^0 \in \mathbb{R} \) satisfies \( F(\lambda_1^0) = 1 \)). Computing numerically this value, we obtain that \( \lambda_1 = 0.239 \). We first choose as initial state an eigenfunction corresponding to \( \lambda_1 \)

\[
p_0(a, x) = \varphi_1(a, x) = \varphi_1^0(a) \varphi_1^D(x) = \frac{a^* - a}{a^*} e^{-\lambda_1^0 a} \sin(x).
\]
The exact solution is:

\[ p(a, x, t) = e^{\lambda_1 t} p_0(a, x). \]

We take: \( T = 2a^* \), with \( a^* = 2 \).

Using \( N_x = 100 \), \( N_a = 120 \) and \( N_t = 2N_a \), we obtain an \( L^2 \) relative error of 4.07%.

Estimated (left) and exact (right) solution at time \( t = T \).
Second test : initial state = Gaussian function

We choose a space-aged localized initial distribution of population of gaussian type:

\[ p_0(a, x) = \exp \left\{ -\left(30\left(a - a^*/4\right)^2 + 20(x - \ell/4)^2\right) \right\}. \]

Gaussian initial state (3D and 2D representations).
We obtain an $L^2$ relative error of 2.99%, 9.6% and 16.2% respectively for 5%, 10% and 15% of noise\textsuperscript{1}.

\textsuperscript{1}“Exact” solution refers here to a numerical solution computed numerically.
Estimated and exact final total population

\[ P_T(x) = \int_0^{a^*} p(a, T, x) \, da \quad \text{and} \quad \hat{P}_T(x) = \int_0^{a^*} \hat{p}(a, T, x) \, da. \]

Estimated (dashed line) and exact total population at time \( t = T \) with 5\% of noise (left) and 15\% of noise (right).
Distributed observation in space and age

Estimated (dashed line) and exact total population at time $t = T$ with age observation in $\left(0, \frac{a^*}{20}\right)$ (left) and $\left(\frac{a^*}{2}, a^*\right)$ (right).
Influence of the observation time

We consider a configuration with two unstable eigenvalues and we investigate the influence of $T$. For $T = 0.5a^*$, we obtain a relative error of 27% for the population density, but we still obtain a reasonable approximation for the total population.

Estimated (dashed line) and exact total population at time $t = T$, for $T = a^*$ (left) and for $T = 0.5a^*$ (right).
Conclusion

## Other models

- **Other outputs**: \( y(x, t) = \int_{a_1}^{a_2} p(a, x, t) \, da, \quad x \in \mathcal{O}. \)
- **Space dependent coefficients**: \( \beta(a, x), \mu(a, x). \)
- **Nonlinearities**: \( \beta(a, x, P) \) and \( \mu(a, x, P) \) where
  \[
  P(x, t) := \int_0^{a_*} p(a, x, t) \, da.
  \]

- Adaptative observer which gives an estimation of \( p \) and \( k. \)

## Approximation

- **Convergence analysis and error estimates**
- **Uniform exponential stability (with respect to \( \Delta a \) et \( h \))**
Particular case: \( A_+ := A|_{D(A) \cap X_+} \) diagonalizable

The results collected here can be found in Barbu and Triggiani 2004. We assume that \( A_+ := A|_{D(A) \cap X_+} \) is diagonalizable. For simplicity, we denote by \( N \) the number of unstable eigenvalues of \( A \) counted with multiplicities (still denoted \( \lambda_k, k = 1, \cdots, N \)). This implies in particular that the unstable space is

\[
X_+ = \bigoplus_{k=1}^{N} \ker(A - \lambda_k).
\]

We denote then by \((\varphi_k)_{1 \leq k \leq N}\) a basis of \( X_+ \). Denote by \( \psi_k \) an eigenfunction of \( A^* \) corresponding to the unstable eigenvalue \( \overline{\lambda_k} \) \( (1 \leq k \leq N) \). It can be shown that the family \((\psi_k)_{1 \leq k \leq N}\) can be chosen such that \((\varphi_k)_{1 \leq k \leq N}\) and \((\psi_k)_{1 \leq k \leq N}\) form bi-orthogonal sequences, in the sense that \((\varphi_k, \psi_m)_X = \delta_{km} \). It follows then that the projection operator \( P_+ \in \mathcal{L}(X, X_+) \) can be expressed as

\[
P_+z = \sum_{k=1}^{N} (z, \psi_k)_X \varphi_k \quad (z \in X).
\]
Since
\[ X_+ = P_+ X = \text{Span} \{ \varphi_k, 1 \leq k \leq N \}, \]
it follows that
\[ Y_+ = C X_+ = \text{Span} \{ C \varphi_k, 1 \leq k \leq N \}. \]

Assume now that the family
\[ (C \varphi_k)_{1 \leq k \leq N} \text{ is linearly independent in } X. \quad (1) \]

This property holds true in the case of internal observation. Therefore
\[ \dim Y_+ = \dim X_+ = N. \]

We denote by \( G \) the Hermitian matrix of size \( N \times N \) defined by
\[ G = \left( (C \varphi_i, C \varphi_j)_Y \right)_{1 \leq i \leq N, 1 \leq j \leq N}. \]

It is not difficult to prove that (1) is equivalent to the fact that \( G \) is invertible.
The orthogonal projection operator $Q_+$

**Lemma**

Assume that property (1) holds true. Then, for any $y \in Y$, then $Q_+ y$ is defined by

$$Q_+ y = \sum_{i=1}^{N} (y, \eta_i) Y C \varphi_i,$$

where

$$\eta_i = \sum_{j=1}^{N} \alpha_{ij} C \varphi_j$$

and

$$(\alpha_{ij})_{1 \leq i \leq N, 1 \leq j \leq N} = G^{-1}.$$
From \( L_+ \) to \( L \)

The (finite dimensional) operator \( C_+ \in \mathcal{L}(X_+, Y_+) \) satisfies \( C_+ \varphi_k = C \varphi_k \) for any \( k \in \{1, \ldots, N\} \). Note that \( C_+ \) is nothing but the identity matrix when we choose as basis for \( X_+ \) and \( Y_+ \) respectively \((\varphi_k)_{1 \leq k \leq N} \) and \((C \varphi_k)_{1 \leq k \leq N}\). Therefore, using these bases, \( A_+ + L_+ C_+ \) is a Hurwitz matrix provided \( \text{diag}(\lambda_1, \ldots, \lambda_N) + L_+ \) is Hurwitz. It is thus sufficient to take \( L_+ = -\sigma I \) with

\[
\sigma > \Re \lambda_1
\]

to ensure the stability of \( A_+ + L_+ C_+ \).

The corresponding operator \( L \in \mathcal{L}(Y, X) \) for every \( y \in Y \)

\[
Ly = L_+ Q_+ y = L_+ \left( \sum_{i=1}^{N} (y, \eta_i)_Y C \varphi_i \right) = -\sigma \sum_{i=1}^{N} (y, \eta_i)_Y \varphi_i,
\]

and, following Theorem 6, \( A + LC \) generates an exponentially stable semigroup.
Under these assumptions, there is a unique unstable eigenvalue
\[ \lambda_1 = \lambda_1^0 - 1 \] (where \( \lambda_1^0 \in \mathbb{R} \) satisfies \( F(\lambda_1^0) = 1 \)). Computing numerically this value, we obtain that \( \lambda_1 = 0.239 \).

The observation operator \( C \in \mathcal{L}(X,Y) \) is given by
\[ C\varphi = \varphi|_{(0,a^*) \times (\pi/3, 2\pi/3)}, \quad \forall \varphi \in X \]
where \( X = L^2((0, a^*) \times (0, \pi)) \) and \( Y = L^2((0, a^*) \times (\pi/3, 2\pi/3)) \).

In order to estimate \( p(T) \), we use the observer designed previously.
As the unstable space is the one-dimensional space
\[ X_+ = \text{Ker} (A - \lambda_1) = \text{Span} \{ \varphi_1 \} = \text{Span} \{ \varphi_1^0(a)\varphi_D^1(x) \}, \]
the observer involves the stabilizing output injection operator \( L \)
defined by
\[ Ly = -\sigma(y, \eta_1)_Y \varphi_1 \quad (y \in Y), \]
where \( \sigma > \lambda_1 \) (gain coefficient) and
\[ \eta_1 = \alpha_{11} C\varphi_1 = \frac{C\varphi_1}{\|C\varphi_1\|_Y^2}. \]
The observer solves then the following system

\[
\begin{aligned}
\partial_t \hat{p}(a, x, t) + \partial_a \hat{p}(a, x, t) + \mu(a) \hat{p}(a, x, t) \\
-\partial_{xx} \hat{p}(a, x, t) + \sigma \left( C \hat{p}, \eta_1 \right)_Y \varphi_1(a, x) \\
= \sigma \left( y, \eta_1 \right)_Y \varphi_1(a, x),
\end{aligned}
\]

\[
\hat{p}(a, 0, t) = \hat{p}(a, \pi, t) = 0,
\]

\[
\hat{p}(a, x, 0) = 0,
\]

\[
\hat{p}(0, x, t) = \int_0^{a^*} \beta(a) \hat{p}(a, x, t) \, da,
\]

\[
\hat{p}(0, x, t) = 0,
\]

\[
x \in (0, \pi), \quad t > 0.
\]

Goal: compare \( p(T) \) and \( \hat{p}(T) \)
Main difficulties concerning the discretization

- Singular behavior of the coefficient $\mu$

  $\leftrightarrow$ rescaling the problem: introduce the auxiliary variable

  $u(a, x, t) = p(a, x, t) = \exp \left( \int_0^{a^*} \mu(s) \, ds \right) p(a, x, t)$

  $\left\{
  \begin{array}{l}
  \partial_t u(a, x, t) + \partial_a u(a, x, t) - \partial_{xx} u(a, x, t) = 0, \\
  u(a, 0, t) = u(a, \pi, t) = 0, \\
  u(a, x, 0) = u_0(a, x) = p_0(a, x)/\Pi(a), \\
  u(0, x, t) = \int_0^{a^*} m(a) u(a, x, t) \, da,
  \end{array}
  \right.$

  where $m(a) = \beta(a) \Pi(a)$

- Discretization of the renewal eq.: $p(0, x, t) = \int_0^{a^*} \beta(a) p(a, x, t) \, da$

  $\leftrightarrow u(0, x, n\Delta t) = \int_0^{a^*} m(a) u(a, x, (n - 1)\Delta t) \, da$

- Presence of the extra term in the observer equation $(C\hat{p}, \eta_1)_Y \varphi_1$,

  $\leftrightarrow$ introduce $\theta(t) = (C\Pi \hat{u}, \eta_1)_Y$ which satisfies

  $\left\{
  \begin{array}{l}
  \dot{\theta}(t) = -(C\Pi \partial_a \hat{u}, \eta_1)_Y + (C\Pi \partial_{xx} \hat{u}, \eta_1)_Y - \sigma \theta(t) + \sigma (y, \eta_1)_Y \\
  \theta(0) = 0
  \end{array}
  \right.$
Rescaling the open loop problem

First of all, in order to overcome the difficulties due to singular behavior of the coefficient $\mu$, we introduce the auxiliary variable

$$u(a, x, t) = \frac{p(a, x, t)}{\Pi(a)} = \exp \left( \int_{0}^{a^*} \mu(s) \, ds \right) p(a, x, t).$$

One can easily check that $u$ satisfies

$$\begin{cases}
\partial_t u(a, x, t) + \partial_a u(a, x, t) - \partial_{xx} u(a, x, t) = 0, & a \in (0, a^*), x \in (0, \pi), t > 0, \\
u(a, 0, t) = u(a, \pi, t) = 0, & a \in (0, a^*), t > 0, \\
u(a, x, 0) = u_0(a, x), & a \in (0, a^*), x \in (0, \pi), \\
u(0, x, t) = \int_{0}^{a^*} m(a) u(a, x, t) \, da, & x \in (0, \pi), t > 0,
\end{cases}$$

where we have set $u_0(a, x) = \frac{p_0(a, x)}{\Pi(a)}$ and where $m(a) = \beta(a) \Pi(a)$ stands for the maternity function.
Let $u^n(a, x)$ be an approximation of $u(a, x, t^n)$, where $t^n = n\Delta t$, $0 \leq n \leq N_t$, $\Delta t = T/N_t$ is a discretization of $(0, T)$. Starting from $u^0(a, x) = u_0(a, x)$, we construct $u^n$ for $n \geq 1$ using an Euler's backwards scheme

$$\frac{u^n(a, x) - u^{n-1}(a, x)}{\Delta t} + \partial_a u^n(a, x) - \partial_{xx} u^n(a, x) = 0, \quad a \in (0, a^*), \ x \in (0, \pi),$$

$$u^n(a, 0) = u^n(a, \pi) = 0, \quad a \in (0, a^*),$$

$$u^0(a, x) = u_0(a, x), \quad a \in (0, a^*), \ x \in (0, \pi),$$

$$u^n(0, x) = \int_0^{a^*} m(a) u^{n-1}(a, x) \, da, \quad x \in (0, \pi).$$
Denoting by $u^n_i(a)$ an approximation of $u^n(x_i, a)$ (where $x_i = ih = i\ell/(N_x + 1)$, with $0 \leq i \leq N_x + 1$) and using a classical centered approximation for the second order derivative in space, the above system yields

$$\left\{ \begin{array}{l}
\frac{dU^n}{da}(a) + \frac{1}{h^2} K U^n(a) + \frac{1}{\Delta t} U^n(a) = \frac{1}{\Delta t} U^{n-1}(a), \\
U^n(0) = \int_{0}^{a^*} m(a) U^{n-1}(a) \, da, \\
U^0(a) = U_0(a),
\end{array} \right.$$ 

where

$$U^n(a) = \begin{pmatrix}
    u^n_1(a) \\
    \vdots \\
    \vdots \\
    u^n_{N_x}(a)
\end{pmatrix}, \quad U_0(a) = \begin{pmatrix}
    u_0(a, x_1) \\
    \vdots \\
    \vdots \\
    u_0(a, x_{N_x})
\end{pmatrix}, \quad K = \begin{pmatrix}
    2 & -1 & & & \\
    -1 & 2 & -1 & 0 \\
    & \ddots & \ddots & \ddots \\
    0 & -1 & 2 & -1 \\
    & & & -1 & 2
\end{pmatrix}.$$
Finite difference discretization in age

We use a Crank-Nicholson scheme. Denoting by $u_{i}^{n,k}$ an approximation of $u_{i}^{n}(a^{k})$, where $a^{k} = k\Delta a$, $0 \leq k \leq N_{a}$, $\Delta t = a^{*}/N_{a}$, and by

$$
U^{n,k} := \begin{pmatrix}
u_{1}^{n,k} \\
\vdots \\
u_{N_{x}}^{n,k}
\end{pmatrix}
$$

an approximation of $U^{n}(a^{k})$, we move from age $a^{k-1}$ to age $a^{k}$ following

$$
\frac{1}{\Delta a} \left( U^{n,k} - U^{n,k-1} \right) + \frac{1}{h^{2}} K \left( \frac{U^{n,k} + U^{n,k-1}}{2} \right) + \frac{1}{\Delta t} \left( \frac{U^{n,k} + U^{n,k-1}}{2} \right) = \frac{1}{\Delta t} \left( \frac{U^{n-1,k} + U^{n-1,k-1}}{2} \right),
$$

with the initial conditions

$$
\begin{cases}
U^{0,k} = U_{0}(a^{k}), & \forall k = 0, \ldots, N_{a}, \\
U^{n,0} = \sum_{k=0}^{N_{a}} \omega_{k} m(a^{k}) U^{n-1,k} \approx \int_{0}^{a^{*}} m(a) U^{n-1}(a) da.
\end{cases}
$$
The algorithm

1. For \( n = 0 \): Initialization of \( \mathbf{U}^{0,k} \)

2. For \( n = 1, \ldots, N_t \):
   - \( k = 0 \): Initialization of \( \mathbf{U}^{n,0} \) using the values of \( (\mathbf{U}^{n-1,j})_{j=0}^{N_a} \).
     \[
     \mathbf{U}^{n,0} = \sum_{k=0}^{N_a} \omega_k m(a^k) \mathbf{U}^{n-1,k}
     \]
   - For \( k = 1, \ldots, N_a \), \( \mathbf{U}^{n,k} = \begin{pmatrix} u_1^{n,k} \\ \vdots \\ u_{N_x}^{n,k} \end{pmatrix} \) solves the linear system
     \[
     \mathbf{A} \mathbf{U}^{n,k} = \mathbf{b}^{n,k}
     \]
     where
     \[
     \mathbf{A} = \left( \Delta t + \frac{1}{2} \Delta a \right) \mathbf{I} + \frac{\Delta t \Delta a}{h^2} \mathbf{K}
     \]
     \[
     \mathbf{b}^{n,k} = \frac{\Delta a}{2} \left( \mathbf{U}^{n-1,k} + \mathbf{U}^{n-1,k-1} \right) + \left[ \left( \Delta t - \frac{\Delta a}{2} \right) \mathbf{I} - \frac{\Delta t \Delta a}{2h^2} \mathbf{K} \right] \mathbf{U}^{n,k-1}
     \]
   - \( \mathbf{Y}^{n,k} = \begin{pmatrix} y_1^{n,k} \\ \vdots \\ y_{N_x}^{n,k} \end{pmatrix} \) where \( y_i^{n,k} = \Pi(a^k)u_i^{n,k} \) if \( \ell_1 \leq ih \leq \ell_2 \) and
Discretization of the closed loop system: observer design

\[
\begin{aligned}
\partial_t \hat{p}(a, x, t) &+ \partial_a \hat{p}(a, x, t) + \mu(a) \hat{p}(a, x, t) \\
- \partial_{xx} \hat{p}(a, x, t) + \sigma (C \hat{p}, \eta_1)_{Y} \varphi_1(a, x) &= \sigma (y, \eta_1)_{Y} \varphi_1(a, x), \\
\hat{p}(a, 0, t) &= \hat{p}(a, \pi, t) = 0, \\
\hat{p}(a, x, 0) &= 0, \\
\hat{p}(0, x, t) &= \int_{0}^{\alpha^*} \beta(a) \hat{p}(a, x, t) \, da.
\end{aligned}
\]
Rescaling the problem

First of all, we introduce the auxiliary variable

\[ \hat{u}(a, x, t) = \frac{\hat{p}(a, x, t)}{\Pi(a)} = \exp \left( \int_0^{a^*} \mu(s) \, ds \right) \hat{p}(a, x, t). \]

One can easily check that \( \hat{u} \) satisfies

\[
\begin{aligned}
\partial_t \hat{u}(a, x, t) + \partial_a \hat{u}(a, x, t) - \partial_{xx} \hat{u}(a, x, t) \\
+ \sigma \left( C \Pi \hat{u}, \eta_1 \right)_Y v_1(a, x) = \sigma \left( y, \eta_1 \right)_Y v_1(a, x),
\end{aligned}
\]

\[
\begin{aligned}
\hat{u}(a, 0, t) = \hat{u}(a, \pi, t) = 0, \\
\hat{u}(a, x, 0) = 0, \\
\hat{u}(0, x, t) = \int_0^{a^*} m(a) \hat{u}(a, x, t) \, da,
\end{aligned}
\]

where we have set \( v_1(a, x) = \varphi_1(a, x)/\Pi(a) \).
Discretization of the term \((C\Pi\hat{u}, \eta_1)_Y\)

Let us introduce

\[ \theta(t) = (C\Pi\hat{u}, \eta_1)_Y. \]

Using the fact that \((C\Pi v_1, \eta_1)_Y = 1\), we remark that \(\theta\) satisfies

\[ \dot{\theta}(t) = - (C\Pi \partial_a \hat{u}, \eta_1)_Y + (C\Pi \partial_{xx} \hat{u}, \eta_1)_Y - \sigma \theta(t) + \sigma (y, \eta_1)_Y. \]

Consequently,

\[
\begin{cases}
\dot{\theta}(t) = - (C\Pi \partial_a \hat{u}(t), \eta_1)_Y + (C\Pi \partial_{xx} \hat{u}(t), \eta_1)_Y - \sigma \theta(t) + \sigma (y(t), \eta_1)_Y, \\
\partial_t \hat{u}(a, x, t) + \partial_a \hat{u}(a, x, t) - \partial_{xx} \hat{u}(a, x, t) + \sigma \theta(t) v_1(a, x) \\
\quad = \sigma (y, \eta_1)_Y v_1(a, x), \\
\theta(0) = 0, \\
\hat{u}(a, 0, t) = \hat{u}(a, \ell, t) = 0, \\
\hat{u}(a, x, 0) = 0, \\
\hat{u}(0, x, t) = \int_0^{a^*} m(a) \hat{u}(a, x, t) \, da.
\end{cases}
\]
Finite difference discretization in time

Let $\hat{u}^n(a,x)$ (resp. $\theta^n, y^n(a,x)$) be an approximation of $\hat{u}(a,x,t^n)$ (resp. $\theta(t^n), y(a,x,t^n)$), where $t^n = n\Delta t$, $0 \leq n \leq N_t$, $\Delta t = T/N_t$ is a discretization of $(0,T)$. Starting from $\theta^0 = 0$ and $\hat{u}^0(a,x) = 0$, we construct $\theta^n$ and $\hat{u}^n$ for $n \geq 1$ using an Euler’s backwards scheme

\[
\begin{align*}
\frac{1}{\Delta t} (\theta^n - \theta^{n-1}) &= - (C\Pi \partial_a \hat{u}^{n-1}, \eta_1)_Y + (C\Pi \partial_{xx} \hat{u}^{n-1}, \eta_1)_Y \\
&\quad - \sigma \theta^n + \sigma (y^n, \eta_1)_Y, \\
\frac{1}{\Delta t} (\hat{u}^n(a,x) - \hat{u}^{n-1}(a,x)) + \partial_a \hat{u}^n(a,x) - \partial_{xx} \hat{u}^n(a,x) + \sigma \theta^n v_1 \\
&\quad = \sigma (y^n, \eta_1)_Y v_1,
\end{align*}
\]

$\theta^0 = 0$, 
$\hat{u}^n(a,0) = \hat{u}^n(a,\ell) = 0$, 
$\hat{u}^0(a,x) = 0$, 
$\hat{u}(a,x) = 0$, 
$\hat{u}^n(0,x) = \int_0^{a^*} m(a) \hat{u}^{n-1}(a,x) \, da$. 

Finite difference discretization in space

Denoting by \( \hat{u}_i^n(a) \) an approximation of \( \hat{u}^n(x_i, a) \), the above system yields

\[
\begin{cases}
\frac{1}{\Delta t} \theta^n = \frac{1}{\Delta t} \theta^{n-1} - \sigma \theta^n - h \left( \int_{0}^{a^*} \Pi(a) \eta_1(a)^T \partial_a \hat{U}^{n-1}(a) \, da \right) \\
- \frac{1}{h} \left( \int_{0}^{a^*} \Pi(a) \eta_1(a)^T K \hat{U}^{n-1}(a) \, da \right) + \sigma h \left( \int_{0}^{a^*} \eta_1(a)^T Y^n(a) \, da \right),
\end{cases}
\]

\[
\frac{d \hat{U}^n}{da}(a) + \frac{1}{h^2} K \hat{U}^n(a) + \frac{1}{\Delta t} \hat{U}^n(a) + \sigma \theta^n V_1(a)
\]

\[
= \frac{1}{\Delta t} \hat{U}^{n-1}(a) + \sigma h \left( \int_{0}^{a^*} \eta_1(a)^T Y^n(a) \, da \right) V_1(a),
\]

\( \theta^0 = 0, \)

\( \hat{U}^n(0) = \int_{0}^{a^*} m(a) \hat{U}^{n-1}(a) \, da, \)

\( \hat{U}^0(a) = 0, \)

where

\[
\hat{U}^n(a) = \begin{pmatrix} \hat{u}_1^n(a) \\ \vdots \\ \hat{u}_{N_x}^n(a) \end{pmatrix}, \quad V_1(a) = \begin{pmatrix} v_1(a, x_1) \\ \vdots \\ v_1(a, x_{N_x}) \end{pmatrix}, \quad \eta_1(a) = \begin{pmatrix} \eta_1(a, x_1) \\ \vdots \\ \eta_1(a, x_{N_x}) \end{pmatrix}.
\]
Finite difference discretization in age

We use a Crank-Nicholson scheme. Denoting by \( \hat{u}_{i}^{n,k} \) an approximation of \( \hat{u}_{i}^{n}(a^k) \), where \( a^k = k\Delta a \), \( 0 \leq k \leq N_a \), \( \Delta t = a^* / N_a \), and by \( \hat{U}_{n,k} := \begin{pmatrix} \hat{u}_{1}^{n,k} \\ \vdots \\ \hat{u}_{N_x}^{n,k} \end{pmatrix} \) an approximation of \( \hat{U}_{n}(a^k) \), we move from age \( a^{k-1} \) to age \( a^k \) following

\[
\frac{1}{\Delta t} \theta^{n} = \frac{1}{\Delta t} \theta^{n-1} - \sigma \theta^{n} - h\Delta a \left( \sum_{k=1}^{N_a} \Pi(k\Delta a) \mathbf{(\eta}^k_1) \mathbf{T} \left( \frac{\hat{U}^{-1,k} - \hat{U}^{-1,k-1}}{\Delta a} \right) \right) \\
- \frac{\Delta a}{h} \left( \sum_{k=1}^{N_a} \Pi(k\Delta a) \mathbf{(\eta}^k_1) \mathbf{T} \mathbf{K}\hat{U}^{-1,k} \right) + \sigma h\Delta a \left( \sum_{k=1}^{N_a} \mathbf{(\eta}^k_1) \mathbf{T} \mathbf{Y}^{n,k} \right),
\]

and

\[
\frac{1}{\Delta a} \left( \hat{U}_{n,k} - \hat{U}_{n,k-1} \right) + \frac{1}{h^2} \mathbf{K} \left( \frac{\hat{U}_{n,k} + \hat{U}_{n,k-1}}{2} \right) + \frac{1}{\Delta t} \left( \frac{\hat{U}_{n,k} + \hat{U}_{n,k-1}}{2} \right) \\
+ \sigma \theta^{n} \mathbf{V}^{k}_1 = \frac{1}{\Delta t} \left( \frac{\hat{U}^{n-1,k} + \hat{U}^{n-1,k-1}}{2} \right) + \sigma h\Delta a \left( \sum_{j=1}^{N_a} \mathbf{(\eta}^j_1) \mathbf{T} \mathbf{Y}^{n,j} \right) \mathbf{V}^{k}_1,
\]
with the initial conditions

\[
\begin{align*}
\theta^0 &= 0, \\
\hat{U}^{0,k} &= 0, \quad \forall k = 0, \ldots, N_a, \\
\hat{U}^{n,0} &= \sum_{k=0}^{N_a} \omega_k m(a^k) \hat{U}^{n-1,k}.
\end{align*}
\]

Here

\[
\mathbf{V}_1^k = \begin{pmatrix}
v_1(k\Delta a, x_1) \\
\vdots \\
v_1(k\Delta a, x_{N_x})
\end{pmatrix}, \quad \mathbf{\eta}_1^k = \begin{pmatrix}
\eta_1(k\Delta a, x_1) \\
\vdots \\
\eta_1(k\Delta a, x_{N_x})
\end{pmatrix}, \quad \mathbf{Y}^{n,k} = \begin{pmatrix}
y^n(k\Delta a, x_1) \\
\vdots \\
y^n(k\Delta a, x_{N_x})
\end{pmatrix}.
\]
1. For \( n = 0 \): Initialization of \( \theta^0 \) and \( \hat{U}^{0,k} \) \((k = 0, \ldots, N_a)\) at 0.

2. For \( n = 1, \ldots, N_t \):
   - Calculate \( \theta^n \) using the values of \( \theta^{n-1} \) and \( (\hat{U}^{n-1,j})_{j=0}^{N_a} \):
     \[
     \theta^n = \frac{1}{1 + \sigma \Delta t} \left( \theta^{n-1} - h \Delta t \sum_{k=1}^{N_a} \Pi(k \Delta a)(\eta_1^k)^T \left( \hat{U}^{n-1,k} - \hat{U}^{n-1,k-1} \right) \right)
     \]
     \[
     - \frac{\Delta a \Delta t}{h} \sum_{k=1}^{N_a} \Pi(k \Delta a)(\eta_1^k)^T \hat{K} \hat{U}^{n-1,k} + \sigma h \Delta a \Delta t \sum_{k=1}^{N_a} (\eta_1^k)^T \hat{Y}^{n,k}
     \]
   - \( k = 0 \): Initialization of \( \hat{U}^{n,0} \) using the values of \( (\hat{U}^{n-1,j})_{j=0}^{N_a} \):
     \[
     \hat{U}^{n,0} = \sum_{k=0}^{N_a} \omega_k m(k \Delta a) \hat{U}^{n-1,k}
     \]
   - For \( k = 1, \ldots, N_a \), solve the linear system
     \[
     A \hat{U}^{n,k} = \hat{b}^{n,k}
     \]
     where
     \[
     \hat{b}^{n,k} = \frac{\Delta a}{2} \left( \hat{U}^{n-1,k} + \hat{U}^{n-1,k-1} \right) + \left[ (\Delta t - \frac{\Delta a}{2})I - \frac{\Delta t \Delta a}{2h^2} \hat{K} \right] \hat{U}^{n,k-1}
     \]
     \[
     - \sigma \Delta a \Delta t \theta^n V_1^k + \sigma h (\Delta a)^2 \Delta t \left( \sum_{j=1}^{N_a} (\eta_1^j)^T \hat{Y}^{n,j} \right) V_1^k
     \]
End of the algorithm

\[ \hat{P}_{n,k} = \begin{pmatrix} \hat{p}_{1}^{n,k} \\ \vdots \\ \hat{p}_{N_{x}}^{n,k} \end{pmatrix} \quad \text{where} \quad \hat{p}_{i}^{n,k} = \prod (a_{k}^{i}) \hat{u}_{i}^{n,k}. \]